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A System of Inverted Nonsmooth Pendula: Modelling an Elderly Person Stepping over an Obstacle

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Abstract

We derive a mechanical model of human motion where an elderly person decides to step over an obstacle rather than avoiding it. Such a decision may be deliberate or forced due to a sudden appearing obstacle in his/her way. The model is represented by a nonautonomous system of ordinary differential equations with discontinuous right hand side. We provide a notion of lateral stability. It is shown that increasing the angle between legs increases stability linearly. This implies that an individual reduces the risk of falling due to stepping over an obstacle by increasing the angle between legs.

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1 Introduction

In an ageing population the number of accidents related to falls of elderly people is typically rising. Current NHS financial costs associated with falls and fall-related accidents in the UK are estimated at more than £2.3 billion a year according to NICE, clinical guideline 161 published in June 2013. It is shown that people aged 65 years and older have the highest risk of falling. It is reported that at least 30% of the people older than 65 years and 50% of the people older than 80 years fall at least once a year. Accidents due to falling do not only create a financial cost, they also incur a human cost such as distress, pain, loss of confidence, loss of independence, injury, and mortality. Injuries due to falls are the most common cause of mortality in the UK for people aged over 75 years. Hence, falls are a common and serious problem for older people.

Stepping over an obstacle is a typical daily life scenario associated with the risk of falling. The process of stepping over an obstacle presents a particularly challenging task for an elderly person. This paper considers the problem where an elderly person decides to step over an obstacle rather avoiding it by walking around it. This decision may be forced due to a suddenly appearing obstacle in his/her way or deliberate. In any scenario, such a decision introduces some risk of falling because the person needs to change the way s/he controls the balance. Accidents related to gait and balance disorders and weaknesses account for 17% of all causes of fall in older people and are the second main cause of falling after "accidents and environmental hazards"[2].

Judging by our everyday experience legged locomotion appears a rather simple task. We hope, walk and run without thinking about it, and yet the interaction between the skeletal system, muscles, tendons and nerves necessary to generate locomotion is quite complex. According to Alexander however, global leg behaviour seems surprisingly simple, suggesting a spring-like behaviour [12]. This spring-like behaviour motivates an elastic model of legged locomotion initially introduced by Blickhan [10]. This model is referred to in the literature as the spring-loaded inverted pendulum or shortly SLIP model. Others have studied this energy-conservative model (cf. [15], and [5]).

The main problem of interest in the context of legged locomotion is stability. For example we want to know whether stable locomotion can be maintained under small perturbations. Seyfarth et al show that the SLIP model for running exhibits a mechanical self-stabilizing property for an appropriate choice of initial conditions, such as velocity, leg stiffness and angle of attack [1]. Blum et al [16] show that the basin of attraction can be enlarged by introducing a

[2]According to a report called "Recurrent Falls" in Patient.co.uk.

control mechanism such as a swing leg control. In their model, variation of leg parameters prior to touchdown compensates for perturbations of ground level and thus, allows to access previously unstable periodic solutions and even further stabilize already stable solutions. Since parameters are held constant during ground contact, the SLIP model with swing leg control remains energy conservative.

The model discussed so far is a purely mechanical model. In order to move towards a biomechanics model requires introducing muscles, tendons, and nerves. For example, muscles mainly have viscoelastic properties which may explain the landing-take-off asymmetry observed in running (cf. [4], and [14]) and hopping [2], a property that is not inherent in the conservative SLIP model with fixed parameters. In addition, these studies show that leg length, i.e. distance between centre of mass and centre of pressure, is larger at take-off than at touchdown. The force length relationships for human running presented in Lipfert [14] also indicate that stiffness decreases during ground contact. This landing is supported by measurements on joint level. There are already a number of studies considering spring-mass models with either variable rest-length (cf. [7],[8]) or variable stiffness (cf. [3],[9]) during contact.

A common approach to improve explanatory and predictive power of the SLIP model is to increase its structural complexity, following the template-anchor concept introduced in the paper by Full and Koditschek [11], e.g. by adding a trunk [6]. Additional structures, however, complicate analysis, and therefore, fundamental insights might be overlooked. In this paper, however, we consider a fundamentally different approach to modelling legged locomotion compared to the SLIP literature which implicitly assumes that legged individuals are sufficiently flexible and for whom walking is second nature. Our model is motivated by observing gait patterns of elderly people who seem less agile and strong compared to a young person. A spring-mass model, hence, seems less likely to explain the gait pattern of interest to us.

The paper is organised as follows: Section two discusses the basic theory. Section three firstly introduces the main assumptions, definitions, and notation of the model. It then derives the main model in three main steps. Section four provides solutions of the nonlinear and linearized version of the model, and discusses robustness of the later. Section five is a conclusion.

2 The Basic Theory

Stepping over an obstacle requires motion of mass m representing the centre of gravity of a person. This motion is described by a second order differential equation. This equation is derived via energy conservation method, where $E_T = E_P + E_K$ states that the total energy E_T of the physical system is determined by the sum of potential energy $E_P = mgh$ and kinetic energy $E_K = \frac{1}{2}mv^2$. h is the height measured between two angular positions of m along the vertical axis and v is the speed of motion of the mass m. Conservation of energy implies $\Delta E_T = 0$. Hence, $mgh = \frac{1}{2}mv^2$, from which we obtain $v = \sqrt{2gh}$. From the formula of the arc length $s = l\gamma$, where l is the length of a leg (cord of pendulum) and γ is the angular displacement it follows that $\frac{d\gamma}{dt} = \frac{1}{l}\sqrt{2gh}$. From the geometry of the pendulum and assuming an initial condition $y_0 = l\cos\gamma_0$ and assuming that after some swing m is at position $y_1 = l\cos\gamma$, it follows that $h = l(\cos\gamma - \cos\gamma_0)$. Substituting h in $\frac{d\gamma}{dt} = \frac{1}{l}\sqrt{2gh}$ we obtain the first integral equation given by $\frac{d\gamma}{dt} = \sqrt{\frac{2g}{l}(\cos\gamma - \cos\gamma_0)}$. By differentiation of the first integral equation, we obtain the second order differential equation of the pendulum.

$$\frac{d^2\gamma}{dt^2} + \frac{g}{l}\sin\gamma = 0. \tag{1}$$

3 The Model

3.1 Definitions, assumptions, and notation

We consider a scenario where there is an obstacle in a person's way [13]. The person decides to step over this obstacle rather avoiding it. Assuming stiff legs, the person periodically shifts his/her balance between the left L and the right R leg. The angle between the legs is denoted by α. It is measured from leg R to L in counter clockwise direction and is held constant during the transition phase. A minimum angle α_{min} is required in order to successfully overcome the obstacle. This information is known from the context of the situation. Let m denote the mass representing the person's centre of gravity. It is connected with the leg R or L depending on which part of the periodic orbit m presently travels. A supporting straight line $\bar{m}$ goes through the centre of mass m at angle $\frac{\alpha}{2}$.

Let $\bar{g}$ be the gravity line perpendicular to the ground going through the centre of gravity represented by mass m. The angle between the gravity line $\bar{g}$ and the leg R, measured from $\bar{g}$ to R in counter clockwise direction, is represented by $\gamma_R \geq 0$. The angle between the gravity line $\bar{g}$ and the leg L, measured from $\bar{g}$ to L in counter clockwise direction, is represented by $\gamma_L \leq 0$. When both legs are on the ground we observe a discontinuity between γ_R and γ_L where γ_R jumps to γ_L or vice versa. Hence, when γ_R changes sign

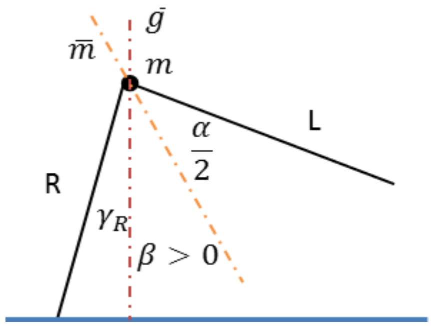


Figure 1: Inverted pendula R, and L

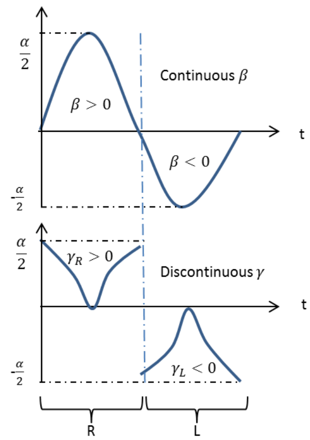


Figure 2: Discontinuous α

its value jumps from $\frac{\alpha}{2}$ to $-\frac{\alpha}{2}$ or vice versa when a change in sign of γ_L is considered. This discontinuity occurs at the point where a switching between the two inverted pendula R and L occurs.

Let the angle between the middle line $\bar{m}$ and the gravity line $\bar{g}$ be denoted by β where β is measured from $\bar{m}$ to $\bar{g}$ in counter clockwise direction. We observe that $\beta > 0$ when m is described by R and $\beta < 0$ when m is described by L. In the case when both legs are on the ground, we observe that $\beta = 0$, and $\gamma_R = \frac{\alpha}{2}$ switches to $\gamma_L = -\frac{\alpha}{2}$ and vice versa depending on the direction of β. At variance to γ_R or γ_L, we observe that β shows no discontinuous behaviour when R switches to L and vice versa.

We now consider the change in direction of β when the person is shifting his/her weight from leg R and L off ground to leg L and R off ground. While shifting the weight on R we observe that $\beta \geq 0$ increases firstly from 0 to $\frac{\alpha}{2}$. On the other hand $\gamma_R \geq 0$ decreases from $\frac{\alpha}{2}$ until it eventually becomes zero. At the right extreme, $\beta \geq 0$ decreases from $\frac{\alpha}{2}$ until it eventually becomes zero again while γ_R increases from zero to $\frac{\alpha}{2}$ again at which point it jumps to γ_L with value $-\frac{\alpha}{2}$. Motion of m on L follows a similar pattern with opposite signs. Hence $\beta \leq 0$ initially decreases to $-\frac{\alpha}{2}$ and then increases to zero again, while γ_L increases from $-\frac{\alpha}{2}$ to zero and then decreases to $-\frac{\alpha}{2}$ again at which point it switches to γ_R. We have described a full oscillation of mass m from R to L to R in terms of α, β, γ_R, and γ_L.

In the next subsections, we derive a mechanical model of an elderly person overcoming an obstacle by oscillating his/her centre of gravity from leg R to

L and back from L to R and so on. The equations of motion describing the trajectories of mass m are those of a system of two inverted pendula. Motion of m on each leg is expressed by a nonautonomous second order ordinary differential equation (ODE). There is also a switching between the inverted pendula R and L. At the switching point, mass m traveling on R continues its journey on L or vice versa depending on its direction. This model requires considering three cases: (I) leg R is on the ground and L in the air, (II) leg L is on the ground and R in the air, and finally (III) both legs are on the ground and switching occurs. The study of a trajectory of mass m requires a model involving all three cases. We derive such a model in the next three subsections.

3.2 Case I: R is on the ground

We consider the movement of mass m on leg R and L off ground. We have

$$\begin{aligned} \gamma_R &< \frac{\alpha}{2} \\ \beta &> 0. \end{aligned}$$

In terms of γ_R, and $0 < \gamma_R < \frac{\alpha}{2}$, we have by equation (1) for $\gamma = \gamma_R$

$$\ddot{\gamma}_R = -\frac{g}{l}\sin(\gamma_R).$$

In terms of β, since $\frac{\alpha}{2} = \gamma_R + \beta$, and $\frac{\alpha}{2} > \beta > 0$, we obtain

$$-\ddot{\beta} = -\frac{g}{l}\sin(\frac{\alpha}{2} - \beta). \tag{2}$$

The two models are equivalent since

$$\ddot{\gamma}_R = -\frac{g}{l}\sin(\gamma_R) = -\frac{g}{l}\sin(\frac{\alpha}{2} - \beta) = -\ddot{\beta},$$

where, $0 < \beta < \frac{\alpha}{2}$, $\frac{\alpha}{2} > \gamma_R > 0$, and $\frac{\alpha}{2} = \gamma_R + \beta$.

3.3 Case II: L is on the ground

We consider the movement of mass m on leg L and R off ground. We have

$$\begin{aligned} \gamma_L &> -\frac{\alpha}{2} \\ \beta &< 0. \end{aligned}$$

In terms of γ_L, and $0 > \gamma_L > -\frac{\alpha}{2}$, we have by equation (1) for $\gamma = -\gamma_L$

$$-\ddot{\gamma}_L = \frac{g}{l}\sin(\gamma_L).$$

In terms of β, since $\frac{\alpha}{2} = \gamma_L - \beta$, and $-\frac{\alpha}{2} < \beta < 0$, we obtain

$$-\ddot{\beta} = \frac{g}{l}\sin(\frac{\alpha}{2} + \beta). \tag{3}$$

The two models are equivalent since

$$-\ddot{\gamma}_L = \frac{g}{l}\sin(\gamma_L) = \frac{g}{l}\sin(\frac{\alpha}{2} + \beta) = -\ddot{\beta},$$

where $0 > \beta > -\frac{\alpha}{2}$, and $-\frac{\alpha}{2} < \gamma_L < 0$.

3.4 Case III: Switching

We now consider the case where the motion of mass m switches from the inverted pendulum R to the inverted pendulum L and vice versa. The system switches at $\beta = 0$. In this position, both legs are on the ground and

$$\ddot{\beta} = \frac{g}{l}\begin{cases} sin(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ sin(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}. \tag{4}$$

Note that this model describes a periodic orbit as the sum of two trajectories, one for each leg. In the first part of the next section we will provide a Lyapunov function and an equation describing the periodic orbits of the model. We then derive a time elapse equation for a simplified model, where $sin(\beta)$ is approximated by β. This linearized model is sufficiently simple but rich in structure in order to derive a simple relationship between ε, β_0 and α, where ε is an exogenous force acting on the model. Stability in the usual sense fails to hold. However, we determine stability in terms of an external force acting on the model. We say that the system is stable under such a perturbation if a perturbation does not shift a current trajectory to a different energy level which is on a trajectory outside a defined separatrix. We then study robustness of the linearized model and derive conclusions.

4 The Time Elapse Equation

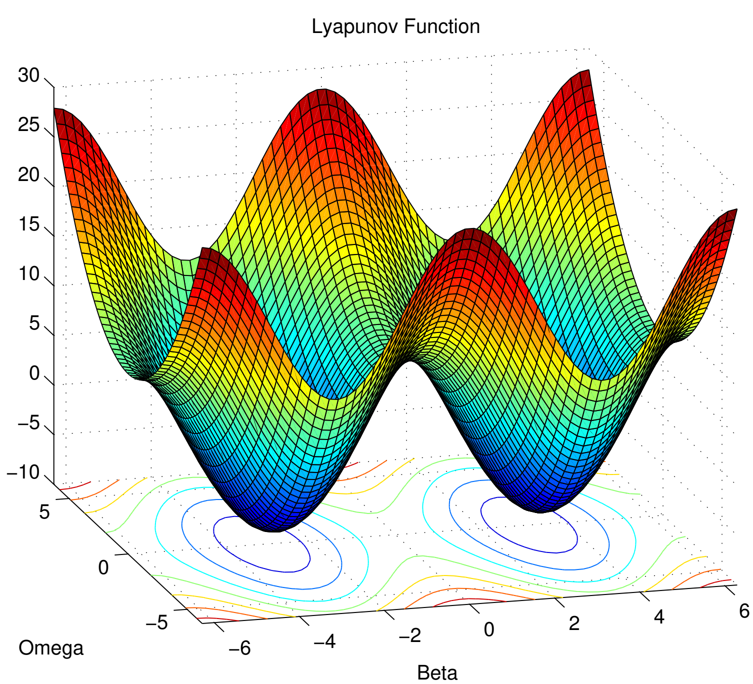
In section three we derived a model (4) depending on the conditions of β, where β is continuous. This is at variance to the model initially depending on γ which is discontinuous. In this section we find a time elapse equation for the linear case, where $sin(\beta) \approx \beta$. Therefore, we first study model (4) by transforming a nonautonomous system of second order ODE's into a system of first order nonautonomous ODE's. We then define a Lyapunov function and derive the equations describing a full periodic orbit. Finally, we provide an equation for the time elapse of a periodic orbit of the linearized model.

From model (4) we obtain a system of first order ODE's

$$\begin{aligned} \dot{\beta} &= \omega \\ \dot{\omega} &= \frac{g}{l}\begin{cases} sin(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ sin(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0. \end{cases} \end{aligned}$$

We define a Lyapunov function V by

$$V(\beta,\omega) := \frac{1}{2}\omega^2 + \frac{g}{l}\begin{cases} cos(\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ cos(\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases} = const = \frac{g}{l}\begin{cases} cos(\beta_0 - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ cos(\beta_0 + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}$$



and using $\frac{d}{dt}V(\beta(t),\omega(t)) = \nabla V \cdot f(\beta(t),\omega(t))$ obtain

$$V'(\beta,\omega) = \begin{cases} \omega \cdot \frac{g}{l} sin(\beta - \frac{\alpha}{2}) - \frac{g}{l} \cdot sin(\beta - \frac{\alpha}{2}) \cdot \omega & \text{, if } \beta \geq 0 \\ \omega \cdot \frac{g}{l} sin(\beta + \frac{\alpha_L}{2}) - \frac{g}{l} \cdot sin(\beta + \frac{\alpha}{2}) \cdot \omega & \text{, if } \beta < 0 \end{cases} = 0.$$

The contour of the Lyapunov function V shows stable and unstable orbits of the system of inverted pendula. These orbits depend on the initial conditions of α, and β_0. In terms of $\omega = \dot{\beta}$, we obtain an equation for the phase paths for fixed values of $C = const$.

$$\omega = \omega_R + \omega_L, \tag{5}$$

where

$$\begin{aligned} \omega_R &= \pm\sqrt{2C - 2\frac{g}{l}cos(\beta - \frac{\alpha}{2})} \text{ if } \beta \geq 0 \\ \omega_L &= \pm\sqrt{2C - 2\frac{g}{l}cos(\beta + \frac{\alpha}{2})} \text{ if } \beta < 0. \end{aligned}$$

The positive and negative values of ω_R together describe the part of the orbit of mass m when leg R is fixed and L off ground. The left leg L contributes to the description of mass m via positive and negative values of ω_L. The picture shows some orbits for different initial conditions represented by the constant C. The picture shows that ω produces unstable orbits for $C \geq \frac{g}{l}$. Such orbits are separatrices and oscillations with no physical relevance to our model. We will show later that we are interested in orbits, which lie inside the separatrix.

4.1 Solution for small angles

When angles are small, then we can consider a linearized version of the model above. Hence, let $sin(\beta) \approx \beta$. In the form of a second order differential equation, we have

$$\ddot{\beta} = \frac{g}{l}\begin{cases} (\beta - \frac{\alpha}{2}) & \text{, if } \beta \geq 0 \\ (\beta + \frac{\alpha}{2}) & \text{, if } \beta < 0 \end{cases}. \tag{6}$$

The homogenous equation is given by

$$\ddot{\beta} - \frac{g}{l}\beta = 0. \tag{7}$$

We can find a solution of this differential equation via characteristic equation. The characteristic equation is given by

$$\lambda^2 = \frac{g}{l}.$$

Hence, $\lambda = \pm\sqrt{\frac{g}{l}}$. The general solution of the homogenous equation (7) is given by

$$\beta(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t},$$

which, with $\beta = \frac{\alpha}{2}$ as a constant becomes

$$\beta(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t} \pm \frac{\alpha}{2},$$

depending on $\beta \geq 0$ or $\beta < 0$. We now solve the initial value problem of a second order differential equation, and use the observation that the solution is a special case since the roots of the characteristic equation satisfy $\lambda_1 = -\lambda_2$. Hence,

$$\begin{aligned} \beta(0) &= \beta_0 < \frac{\alpha}{2} \\ &\Rightarrow c_1 + c_2 + \frac{\alpha}{2} = \beta_0, \end{aligned}$$

and

$$\begin{aligned} \dot{\beta}(0) &= \sqrt{\frac{g}{l}}(c_1 - c_2) = 0 \\ &\Rightarrow c_1 = c_2 = \frac{\beta_0 - \frac{\alpha}{2}}{2}. \end{aligned}$$

Hence, in follows that

$$\beta(t) = \left(\beta_0 \mp \frac{\alpha}{2}\right) cosh\left(\sqrt{\frac{g}{l}}t\right) \pm \frac{\alpha}{2}.$$

The next step requires to use the formula for the time interval of an orbit. From

$$0 = \left(\beta_0 - \frac{\alpha}{2}\right) cosh\left(\sqrt{\frac{g}{l}}t\right) + \frac{\alpha}{2}$$

we obtain for $\beta_0 \to 0$

$$\begin{aligned} cosh\left(\sqrt{\frac{g}{l}}t\right) &= \frac{\frac{-\alpha}{2}}{\beta_0 - \frac{\alpha}{2}} \\ &= \frac{1}{-\frac{2}{\alpha}\beta_0 + 1} \\ t &= \frac{arcosh\left(\frac{1}{1-2\frac{\beta_0}{\alpha}}\right)}{\sqrt{\frac{g}{l}}}. \end{aligned}$$

The formula for time T of a full period orbit of the linearized model is obtain by considering a full oscillation, hence 4 times t, which then becomes

$$T = \frac{4}{\sqrt{\frac{g}{l}}} arcosh \left(\frac{1}{1 - 2\frac{\beta_0}{\alpha}} \right). \tag{8}$$

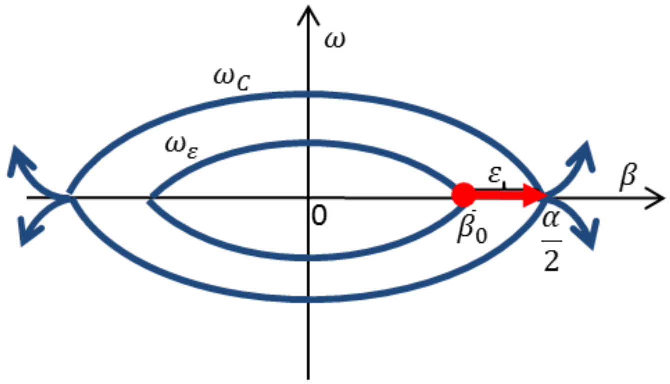
We apply equation (8) in the characterization of robustness of our model. Intuitively, we expect that for a fixed value of T, a small increase in α increases β_0 proportionally. Consequently we expect the region of stable orbits to increase for a proportional increase in both parameters. In the next section we will define a notion of stability and show robustness of the linearized model. Essentially there are three key ideas involved in demonstrating robustness. First, a periodic orbit is stable if it lies inside a defined separatrix. This is a property of the Lyapunov function. We provide the conditions on β_0 and α producing this separatrix. Second, we define an external force $\varepsilon(\beta_0, \alpha)$ acting on the model. Associated with this force, we define an unique stable periodic orbit, ω_ε. We then characterize all stable β_0. These β_0 produce unique orbits inside ω_ε satisfying the perturbation conditions. Finally, we demonstrate robustness of our model by showing the effects of changes in α on β_0 and ε. The effects of a change in β_0 on α and ε are also evident from the proof.

4.2 Stability and robustness of the linearized model

We want to show robustness of or model in terms of changes in α. To show this we progress along three steps.

(1) We define for a fixed value of α its associated separatrix ω_C. The choice of α satisfies $\alpha \geq \alpha_{min}$, where α_{min} is the minimum angle require in order to successfully overcome an obstacle. (2) We then pick the unique stable periodic orbit ω_ε through $\overline{\beta_0}$ which lies inside the separatrix. This orbit is laterally stable at $\overline{\beta_0}$ because $\overline{\beta_0} + \varepsilon(\alpha, \overline{\beta_0})$ is another orbit inside the separatrix, where $\varepsilon(\alpha, \overline{\beta_0})$ is an exogenously determined perturbation through the choice of $\alpha, \overline{\beta_0}$. We characterize all stable β_0 associated with α and $\overline{\beta_0}$. (3) We apply the time elapse equation (8) of the linearized model to show the effect of a change in α from α to α_{new} on $\overline{\beta_0}$. This relation is then used to show robustness in terms of $\varepsilon(T, \alpha_{new})$ for fixed T.

We can determine using equation (5) periodic orbits for different values of C. We are interested in values of C which satisfy $\frac{g}{l} > C > -\frac{g}{l}$. For such values of C, we know that periodic orbits are stable as they are inside the orbit defining a separatrix. Now, let's consider the separatrix, where $C = \frac{g}{l}$ must



be satisfied. We assume that

$$C = \frac{g}{l} cos\left(\beta_0 \mp \frac{\alpha}{2}\right).$$

Then it must be that, since for

$$C = \frac{g}{l} cos\left(\beta_0 \mp \frac{\alpha}{2}\right) = \frac{g}{l}$$

$$\begin{aligned} cos\left(\beta_0 \mp \frac{\alpha}{2}\right) &= 1 \\ \beta_0 \mp \frac{\alpha}{2} &= 0 \\ \beta_0 &= \pm\frac{\alpha}{2}. \end{aligned}$$

Step 1: For a fixed value of α, we determine the separatrix ω_C via equation (8) . For all $\beta_0 < \frac{\alpha}{2}$ we know from the properties of the Lyapunov function that C is such that (8) produces orbits inside the separatrix, which hence are stable by properties of the pendulum. Note that the choice of α is such that $\alpha \geq \alpha_{min}$ where α_{min} is the minimum angle between R and L required in order for a person to successfully step over a given obstacle.It is assumed known from the context of the situation.

Step 2: We now also fix $\overline{\beta_0}$ and define stability in terms of an external force ε acting on the model. We formulate ε in terms of parameters $\overline{\beta_0}$, and $\frac{\alpha}{2}$ by

$$\varepsilon\left(\overline{\beta_0}, \frac{\alpha}{2}\right),$$

where

$$\varepsilon = \frac{\alpha}{2} - \overline{\beta_0}.$$

This notion of stability considers the case where an external force ε acts on mass m at point $\overline{\beta_0}$ in direction $\beta > \overline{\beta_0}$ when motion of mass m is at the right extreme (or $\beta < \overline{\beta_0}$ for left extreme) of the periodic orbit. Then the system is stable subject to a perturbation ε for all β_0 satisfying $\beta_0 + \varepsilon < \frac{\alpha}{2}$. For a perturbation ε we observe that the system is stable for all $\beta_0 \in (0, \overline{\beta_0})$[3]. The periodic orbit ω_ε associated with α and $\overline{\beta_0}$ is given by equation (8).

Step 3: We now want to show that a change in α from α to α_{new} affects the stability interval $(0, \overline{\beta_0})$ and associated robustness interval ε given by $(\overline{\beta_0}, \frac{\alpha}{2})$. Hence, in addition to α and $\overline{\beta_0}$ we also fix T in equation (8). Then for any $K \in \mathbb{R}$ we obtain

$$\begin{aligned} T &= \frac{4}{\sqrt{\frac{g}{l}}} arcosh \left(\frac{1}{1 - 2\frac{K \cdot \overline{\beta_0}}{K \cdot \alpha}} \right) \\ &= \frac{4}{\sqrt{\frac{g}{l}}} arcosh \left(\frac{1}{1 - 2\frac{\overline{\beta_0}}{\alpha}} \right) \\ &= \frac{4}{\sqrt{\frac{g}{l}}} \left(arcosh \frac{1}{(1-k)} \right). \end{aligned}$$

From this we directly observe that $\frac{\overline{\beta_0}}{\frac{\alpha}{2}} = k$, where $k = constant$. This yields

$$\overline{\beta_0} = k \cdot \frac{\alpha}{2}, k \in (0, 1).$$

We can now reformulate $\varepsilon(\alpha, \overline{\beta_0})$ in terms of $\varepsilon(\alpha, k)$ which becomes

$$\varepsilon(\alpha, k) = \frac{\alpha}{2} - k\frac{\alpha}{2} = \frac{\alpha}{2}(1 - k).$$

We have shown that T in (8) is invariant for any constant K. Hence, for fixed T let

$$\begin{aligned} \frac{\alpha_{new}}{2} &:= K \cdot \frac{\alpha}{2} \\ \overline{\beta_{0,new}} &:= K \cdot \overline{\beta_0}. \end{aligned}$$

Then

$$k = \frac{2 \cdot K \cdot \overline{\beta_0}}{K \cdot \alpha} = \frac{2 \cdot \overline{\beta_{0,new}}}{\alpha_{new}} = k(T).$$

[3]Note that the system is also stable when a force ε acts on m at β_0 and $\beta < \beta_0$, when motion on leg L is considered.

Robustness then follows from

$$\varepsilon(\alpha, T) = \frac{\alpha}{2}(1 - k(T)), \text{ for } k(T) \in (0, 1).$$

We have shown that robustness of our model is a linear relationship between ε and α. Increasing α increases robustness ε.

5 Conclusion

This paper considers the situation where an elderly person decides to step over an obstacle rather than avoiding it. This may be a forced decision due to a suddenly appearing obstacle in his/her way or a deliberate decision. In either case, this is a daily life situation potentially leading to accidents due to the risk of falling. Associated with such accidents are personal suffering, private, and financial costs.

This paper develops a mechanical model of human motion and addresses the problem of lateral stability. A stability robustness condition leading to a reduction of risk of falling of elderly people is derived from a system of nonautonomous ordinary differential equations with discontinuous right hand side. The new insights obtained in this paper may help physiotherapists and physicians to educate elderly people about gait strategies to overcome obstacles. Future work should empirically verify the predictive power of this model. This could be done in an experimental setting or via field experiment. This is work in progress.

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