MATHEMATICAL AWARENESS IN LOWER ATTAINING PRIMARY PUPILS: ENABLING PRODUCTIVE MATHEMATICAL CONTRIBUTIONS TO THE PROGRESS OF MIXED ATTAINMENT PAIRS

NANCY JANE BARCLAY

DOCTOR OF EDUCATION

2018
MATHEMATICAL AWARENESS IN LOWER ATTAINING PRIMARY PUPILS: ENABLING PRODUCTIVE MATHEMATICAL CONTRIBUTIONS TO THE PROGRESS OF MIXED ATTAINMENT PAIRS

A thesis submitted in partial fulfilment of the requirements of the University of Brighton for the degree of Doctor of Education

NANCY JANE BARCLAY

OCTOBER 2018
Abstract

This thesis examines how a focus on mathematical awareness can be harnessed to enable lower attaining primary school pupils to make mathematically valid and productive contributions to mixed attainment pair working. In this thesis, mathematical awareness refers to the detection of relevant mathematical features or relations and is posited as a necessary but not sufficient condition for mathematical reasoning. Literature indicates that prevailing ‘ability’ group structures in the primary mathematics classroom operate to the detriment of lower attainers in terms of their mathematical progress and their self-image. Whilst there are indications that mixed attainment groups benefit the lower attainer through access to the thinking of their higher attaining partners, this area is not well researched. Existing research does not address the independent contribution that lower attainers make to such mixed groupings. A separate body of research identifies the development of authority relations in such mixed groupings which can disadvantage lower attainers.

Drawing on the methodology of Education Design Research, three experienced primary class teachers worked with the researcher to design and enact a classroom-based intervention which promoted mathematical awareness through task structure, the use of enactive representation, teacher emphasis and teacher questioning. Video recording captured the action and interaction of two mixed attainment pairs in each classroom over the course of one preliminary and three research lessons. Each pair of pupils comprised one pupil operating above, and one operating below, age-related expectations. Data analysis focused on critical sequences from within the data corpus in which changes in observed behaviour enabled inference of a new awareness.

The thesis contributes evidence of lower attaining pupils’ mathematical contribution to mixed attainment pairs: the focus on awareness was associated with lower attainers’ awareness of mathematical structure making significant contributions to task progress. The focus on awareness had a more limited impact on lower attainers’ contributions through awareness of mathematical property. Awareness pathways for each lower attainer reveal how the focus on awareness was also associated with changes in the nature of pair interaction. Findings indicate that this was instrumental in enabling lower attainers’ mathematical awarenesses to impact on task progress. The thesis concludes that the focus on awareness supported the provocation of structural awareness and shifted the focus of the task from solution finding to noticing, enabling more equitable engagement.
## Contents

Abstract.............................................................................................................................................. 1

Contents................................................................................................................................................ 2

List of Figures ....................................................................................................................................... 5

List of data extracts .............................................................................................................................. 6

List of Tables ......................................................................................................................................... 7

List of abbreviations .............................................................................................................................. 8

Acknowledgements .............................................................................................................................. 9

Declaration ........................................................................................................................................... 10

Chapter 1 Introduction ......................................................................................................................... 11

Chapter 2 Literature review ................................................................................................................. 16

2.1 Meeting the mathematical learning needs of lower attaining pupils ........................................... 16

2.1.1 Defining lower attainment ......................................................................................................... 17

2.1.2 Provision based on notions of ‘ability’ ......................................................................................... 19

2.1.3 Lower attaining pupils can, and do, think mathematically ..................................................... 22

2.1.4 The potential of mixed attainment groupings .......................................................................... 25

2.1.5 Summary and implications for the current study ..................................................................... 29

2.2 Awareness and Reasoning in pupils’ mathematics learning ....................................................... 30

2.2.1 The development of pupils’ mathematical reasoning ................................................................. 32

2.2.2 Positioning mathematical awareness in relation to mathematical reasoning ........................... 34

2.2.3 Awareness: Exploring related constructs .................................................................................. 35

2.2.4 Awareness and mathematics learning ....................................................................................... 39

2.2.5 Awareness and the lower attainer .............................................................................................. 44

2.2.6 Researching awareness: implications for the current study ..................................................... 46

2.3 Authority and influence in mathematics learning ......................................................................... 50

2.3.1 Theoretical constructs in exploring authority relations ............................................................... 51

2.3.2 Mathematics ‘discipline’ authority relations and pupils’ mathematical learning ................... 52

2.3.3 Mathematics ‘expertise’ authority relations and pupils’ mathematical learning ..................... 55

2.3.4 Group interaction and pupils’ mathematical learning ................................................................. 59

2.3.5 Summary and implications for the current study ..................................................................... 63

2.4 Aims and Research questions ........................................................................................................ 65

2.4.1 Research aims ............................................................................................................................. 66

2.4.2 Research questions ..................................................................................................................... 67

Chapter 3 Methodology ....................................................................................................................... 69

3.1 Ontological perspective .................................................................................................................... 69
3.2 Critical Realism and the social world of the classroom ........................................ 72
3.3 Education Design Research .................................................................................. 74
3.4 Study design .......................................................................................................... 76
3.5 Phase 1: Preparation .............................................................................................. 80
3.6 Phase 2: Iterative Intervention ............................................................................... 82
3.6.1 The intervention: a pedagogical focus on noticing .......................................... 83
3.6.2 Workshops ......................................................................................................... 84
3.6.3 Research lessons ................................................................................................. 86
3.6.4 Data collection activities and processes ............................................................ 87
3.6.5 Lesson activity synopses ................................................................................. 89
3.7 Phase 3: Analysis .................................................................................................. 93
3.7.1 Analysis Stage 1: Description ........................................................................... 95
3.7.2 Analysis Stage 2: Identification and collation of awareness events ................ 97
3.7.3 Analytic frameworks ......................................................................................... 100
3.8 Ethical considerations ............................................................................................ 103
3.8.1 Recruitment and consent: schools and teacher participants ............................ 103
3.8.2 Recruitment and consent: focus pupils ............................................................... 106
3.8.3 Data security and participant anonymity .......................................................... 108
3.8.4 The risk of harm ............................................................................................... 109
3.9 Chapter summary .................................................................................................. 110

Chapter 4 The impact of the focus on noticing on lower attaining pupils’ mathematical awareness ........................................................................................................ 111
4.1 Awareness of structure and generality ................................................................. 113
4.1.1 Research lesson 1 .............................................................................................. 113
4.1.2 Research lesson 2 .............................................................................................. 122
4.1.3 Research lesson 3 .............................................................................................. 130
4.1.4 The preliminary lesson ...................................................................................... 138
4.1.5 Section conclusions ........................................................................................... 143
4.2 Awareness of mathematical pattern and property ............................................... 146
4.2.1 Research lesson 1 .............................................................................................. 146
4.2.2 Research lesson 3 .............................................................................................. 148
4.2.3 The preliminary lesson ...................................................................................... 150
4.2.4 Section discussion and conclusions ................................................................ 151

Chapter 5 The impact of the focus on noticing on authority relations .......... 156
5.1 Authority and influence in the preliminary lesson ............................................. 157
5.1.1 Distribution of authority .................................................................................. 157
5.1.2 Response to awareness of mathematical process.................................. 159
5.1.3 Response to awareness of structure and generality ............................. 163
5.2 Authority and influence in the research lessons ....................................... 168
  5.2.1 Distribution of authority ...................................................................... 168
  5.2.2 Research lesson 1: response to awareness of process ............................. 170
  5.2.3 Research lesson 1: response to awareness of property .......................... 172
  5.2.4 Research lesson 1: response to awareness of structure ......................... 173
  5.2.5 Research lesson 3: response to awareness of structure ......................... 174
  5.2.6 Research lesson 2: a partial counter example ........................................ 177
5.3 Chapter summary ...................................................................................... 179

Chapter 6 Discussion ...................................................................................... 181
  6.1 Awareness and the contribution of lower attainers to mixed pair progress .... 182
  6.2 Authority relations and the focus on awareness ........................................ 187
    6.2.1 Distribution of authority through setting of task goals ......................... 188
    6.2.2 The nature of pair interaction .............................................................. 190
    6.2.3 Access to and use of representations .................................................. 193
    6.2.4 Impact of lower attainers’ mathematical awarenesses ............................ 195
    6.2.5 The nature and affordances of teacher questions ................................. 198

Chapter 7 Conclusions and recommendations .............................................. 201
  7.1 Contributions to knowledge and understanding ....................................... 201
  7.2 Recommendations ................................................................................... 202
  7.3 Limitations of the research ..................................................................... 204
  7.4 Further research ..................................................................................... 205
  7.5 Personal learning and reflection .............................................................. 207
References ..................................................................................................... 209

Appendices .................................................................................................... 225
  Appendix A analysis of task affordances for noticing .................................... 225
  Appendix B Data sample: detailed description .............................................. 230
  Appendix C: Critical sequence transcript .................................................... 231
  Appendix D: Project information sheet ......................................................... 233
  Appendix E: Teacher consent form ............................................................... 236
  Appendix F: Parent/carer information sheet and consent form ..................... 237
  Appendix G Pupil, school and teacher profiles ............................................ 240
List of Figures

Figure 3-1 Research phases and activities .......................................................... 79
Figure 3-2 The dice train ..................................................................................... 90
Figure 3-3 Magic V ............................................................................................ 91
Figure 3-4 What numbers can we make? ............................................................. 92
Figure 4-1 The V as two lines of three numbers ................................................. 114
Figure 4-2 The V as two arm pairs plus a shared base number ....................... 115
Figure 4-3 Joe finds pairs with same arm total first ......................................... 116
Figure 4-4 Joe’s awareness pathway RL1 .......................................................... 117
Figure 4-5 Mia’s awareness pathway RL1 .......................................................... 118
Figure 4-6 Mia indicates the arm pair as she sums to 6 ...................................... 119
Figure 4-7 Mia justifies her solution ................................................................. 120
Figure 4-8 Awareness pathways for Joe and Mia, together with a generalised pathway ... 121
Figure 4-9 Rose selects the 15 ......................................................................... 125
Figure 4-10 Awareness pathway for Rose and Mia together with generalised pathway,... 129
Figure 4-11 Representing the multiple of 3 property .......................................... 130
Figure 4-12 Rose partitions the 7 into 6 plus 1 ............................................... 131
Figure 4-13 Three fours each represented as a three remainder 1 ...................... 132
Figure 4-14 Mia adds another 3 piece to the three remainders ....................... 133
Figure 4-15 The three remainders .................................................................. 134
Figure 4-16 Awareness pathway for all children together with generalised pathway ... 136
Figure 4-17 Awareness pathway Joe and Mia together with generalised pathway PL ...... 142
Figure 4-18 Generalised awareness pathways from RL1,2,3 and preliminary lesson .... 144
Figure 4-19 RL3 recording sheet for Mia and Ruby .......................................... 149
Figure 4-20 RL3 recording sheet for Rose and Hannah ..................................... 150
Figure 4-21 Numicon representation of numbers 1-5 ...................................... 154
Figure 5-1 Ruby completes the train that Mia started .................................... 160
Figure 5-2 Ash works on an impossible arrangement ..................................... 164
Figure 5-3 Ruby works on an impossible arrangement .................................. 165
Figure 5-4 Mia secures the position of the 1 card ............................................ 170
Figure 5-5 Mia is not afforded spatial privilege .............................................. 174
Figure 5-6 Mia is afforded spatial privilege ...................................................... 174
Figure 6-1 Influences operating on the development and use of awarenesses ........ 190
Figure 6-2 Influences operating in the preliminary lesson ............................... 197
Figure 6-3 Influences operating in the research lessons ................................. 197
List of data extracts
Extract 4-1.........................................................................................................................115
Extract 4-2..........................................................................................................................118
Extract 4-3..........................................................................................................................119
Extract 4-4..........................................................................................................................120
Extract 4-5..........................................................................................................................120
Extract 4-6..........................................................................................................................123
Extract 4-7..........................................................................................................................124
Extract 4-8..........................................................................................................................126
Extract 4-9..........................................................................................................................126
Extract 4-10.........................................................................................................................131
Extract 4-11.........................................................................................................................133
Extract 4-12.........................................................................................................................137
Extract 4-13.........................................................................................................................137
Extract 4-14.........................................................................................................................137
Extract 4-15.........................................................................................................................139
Extract 4-16.........................................................................................................................139
Extract 4-17.........................................................................................................................140
Extract 4-18.........................................................................................................................140
Extract 4-19.........................................................................................................................140
Extract 4-20.........................................................................................................................141
Extract 4-21.........................................................................................................................141
Extract 4-22.........................................................................................................................141
Extract 4-23.........................................................................................................................145
Extract 4-24.........................................................................................................................145
Extract 4-25.........................................................................................................................147
Extract 4-26.........................................................................................................................148
Extract 4-27.........................................................................................................................149
Extract 4-28.........................................................................................................................150
Extract 4-29.........................................................................................................................151
Extract 4-30.........................................................................................................................152
Extract 4-31.........................................................................................................................153
Extract 4-32.........................................................................................................................153
Extract 4-33.........................................................................................................................153
Extract 5-1..........................................................................................................................157
Extract 5-2..........................................................................................................................157
Extract 5-3..........................................................................................................................158
Extract 5-4..........................................................................................................................158
Extract 5-5..........................................................................................................................159
Extract 5-6..........................................................................................................................162
Extract 5-7..........................................................................................................................162
Extract 5-8..........................................................................................................................163
Extract 5-9..........................................................................................................................164
Extract 5-10.........................................................................................................................168
Extract 5-11.........................................................................................................................169
Extract 5-12.........................................................................................................................169
Extract 5-13.........................................................................................................................169
For ease of readability some data extracts first presented in chapter 4 are re-presented for further analysis in chapter 5. Where this is the case, the re-presentation is noted in the text and a chapter 5 data extract reference given to the extract used.

List of Tables
Table 3-1 Timeline of school-based preparatory activities .................................................. 81
Table 3-2 Analytic framework for mathematical awarenesses .............................................. 101
Table 3-3 Analytic framework for authority and influence .................................................. 102
Table 4-1 Data availability ..................................................................................................... 112
Table 5-1 Authority and influence in the preliminary lesson .............................................. 167
Table 5-2 Authority and influence in the research lessons .................................................. 179

Note
All figures and tables are in colour
List of abbreviations

BERA: British Educational Research Association
DBRC: Design Based Research Collective
DfE: Department for Education
GCSE: General Certificate of Secondary Education
HA: Higher attainer
KS1: Key stage 1 (school years 1 and 2 for pupils aged 5 to 7 in England)
KS2: Key stage 2 (school years 3-6 for pupils aged 7 to 11 in England)
KS3: Key stage 3 (school years 7-9 for pupils aged 11 to 14 in England)
KS4: Key stage 4 (school years 10-11 for pupils aged 15 to 16 in England)
LA: Lower attainer
MaST: Primary Mathematics Specialist Teacher
NCETM: National Centre for Excellence in Teaching Mathematics
PL: Preliminary lesson
RL: Research lesson
SoE: School of Education at the University of Brighton
UK: United Kingdom
US: United States
Acknowledgements

My thanks go first to all the pupils in the three classrooms in which this study was undertaken for accepting me into their classrooms and including me in their mathematical learning activities. Particular thanks go to those who featured in video sequences for their willingness to engage in challenging mathematical thinking in the presence of both a researcher and of recording equipment. Their enthusiasm for mathematics learning has been an inspiration.

My very sincere thanks go to the three primary school teachers who gave their time, energy and pedagogical wisdom to the research over the course of a year. Their willingness to open up their classrooms and their mathematical practice is remarkable, and I thank them for the opportunity to work alongside them and to learn so much from them.

I have been more fortunate than I could expect in the supervisory team of Dr Nadia Edmond and Dr Els de Geest who have supported me so comprehensively during my doctoral study. I have benefitted from their considerable knowledge and expertise in every aspect of this research and have particularly valued their calm, reflective and constructive approach. Their continued faith in me and genuine interest in and enthusiasm for my research has meant a great deal.

Lastly, I acknowledge with love and great thanks, the ongoing support and encouragement of my family who have made space and time for me to study and have done so without complaint for some considerable time.
Declaration

I declare that the research contained in this thesis, unless otherwise formally indicated within the text, is the original work of the author. The thesis has not been previously submitted to this or any other university for a degree, and does not incorporate any material already submitted for a degree.

Signed

Dated
Chapter 1: Introduction

The research upon which this thesis reports and from which its conclusions and recommendations are drawn is founded on a long-held belief that primary education can and should support the mathematics learning of lower attaining children better than is currently the case. The origins of this belief date from earlier employment in the probation service, where the impact of unsuccessful learning experiences, many of which began in primary school, were evident in the life stories of teenagers convicted of a range of offences. Later experiences both as a primary classroom teacher and subsequently as a local authority primary mathematics consultant offered a progressively broader perspective on the typically limited nature of mathematics learning opportunities for lower attainers. That mathematics learning for some pupils is unsuccessful (as defined by government expectations) is reflected in national datasets: a sizeable proportion of primary aged children do not achieve expected standards by the end of their primary phase of schooling (DfE, 2017). The impact of this lower attainment at primary school is significant. In the main these children fare less well than their peers in the subsequent stage of their mathematics learning in secondary education and have a reduced chance of achieving a good GCSE outcome at the end of KS4 (DfE, 2016a).

I view this as neither acceptable, nor irredeemable. My views on the potential of these children to learn mathematics align with those expressed by Gattegno (1988). Gattegno notes that the learning of the mother tongue takes place without formal tuition, drawing only on pupils’ capacity to use of a set of natural powers possessed by us all. Whilst lists of these powers vary between authors, they include the capacity to abstract and to make transformations, to stress and ignore, identify similarity and difference, to categorise, make trials, to conjecture and to generalise (Gattegno, 2010; Mason, 2008c; Mason and Johnston-Wilder, 2004). Building on this, Gattegno (1988) argues that the study of mathematics requires these same powers, thus if learning is geared towards pupils’ awareness and use of these natural learning capacities the subject should be accessible to all primary aged pupils.
This research draws on Gattegno’s argument that rendering mathematics accessible requires the education of awareness (1989; 1988; 1987). In particular Gattegno (2010) emphasises the process of ‘stressing and ignoring’ as a natural human functioning, without which ‘we cannot see anything’ (2010, p16). On a general level, the selection of particular features for focus, whilst others are diminished in the attention they receive enables us to cope with a potential overload of stimuli and to maintain concentration. However, in the mathematics classroom this facility is particularly important in that it enables the identification of sameness amongst difference and the extraction of underlying pattern. For example, it enables us to identify that, from a set of apparently disparate numbers, we can identify a group that share a ‘sameness’ in that they are all multiples of a particular number. Only when we become aware of such commonality can we act on it.

Indeed, it is the purpose to which mathematical awarenesses may be put that provides the rationale for my focus on this aspect of mathematical learning. Mason (2002, p7) observes that ‘what you do not notice, you cannot act upon’; whilst this comment was made in relation to teachers’ noticing, it applies equally to pupils’ mathematical noticing or awareness. I view mathematical awareness as a necessary, but not a sufficient condition for mathematical reasoning to take place. As will be shown through analysis of literature in chapter 2, reasoning is widely considered to be central to the learning of mathematics (e.g. Yankelewitz et al., 2010; Mueller, 2009; Ball and Bass, 2003), thus a focus on awareness as a necessary pre-condition for reasoning is merited because of the importance of reasoning. That the role of mathematical awareness, or noticing, in pupils’ mathematical learning is currently under-researched (Lobato et al., 2013), further strengthens the rationale for its centrality in the current study.

I also consider that pupils’ learning of mathematics cannot be separated from the social situation in which learning takes place. I view the social world of the classroom not just as the backdrop against which learning occurs, but as intrinsic to what is learned and how. This view is embodied in the ontological perspective I adopt in this study: the critical realist perspective views social structures as real and generative of events within the classroom (Bhaskar, 2008; 1998; Danermark, 2002; Collier, 1994).
In the social world of the primary mathematics classroom, lower attaining pupils face additional challenges. Organisational structures such as ability labelling frequently diminish their mathematical learning opportunities (Marks, 2011). Being positioned as mathematically inferior by both peers and teachers compounds these limitations and frequently contributes to the development of negative self-images (Black, 2004). This research focuses on the mathematical learning of lower attaining pupils who are working with higher attaining partners; literature considered in chapter 2 identifies this as a productive organisational structure but not without its complexities. Since I view the social structures embedded in the classroom relations as central to the learning that ensues, I focus close attention on the interaction between the focus lower attainers and their partners. This supports analysis of how the lower attaining pupil’s awarenesses are received and drawn on in the collaborative mathematical task in which the pair are engaged.

In particular, and building on previous class-based observations, I focus attention on authority relations between pupils as they work together on their mathematical tasks. Whilst the potential for mathematical gains for pupils working in mixed groups is documented (Boaler and Staples, 2008), the potentially negative impact of status differences on the nature and quality of interaction between the pair is also established in research literature (Langer-Osuna, 2016). Literature examined in Chapter 2 identifies that such authority relations frequently disadvantage lower attainers in part due to the status afforded to higher attainers’ contributions regardless of their validity (M. Barnes, 2005).

Earlier phases of my doctoral study have offered opportunity to observe lower attaining pupils during their mathematical activity as they worked with higher attaining peers. These observations provided illustrations of the role of awareness, and of how the response by the higher attainer to the expressed awareness of the lower attainer impacted on the subsequent development of the task. In the following two separate incidents, the mathematical awareness of the same lower attaining pupil led to very different outcomes for him and his partner. In the first, part way through a task, the lower attainer noticed a relevant numerical relationship and was able to reason about this relationship and productively change the way in which he
and his partner engaged with the numbers that they were manipulating; solutions were subsequently found more efficiently. However, in the second his identification of a relevant property of number was repeatedly ignored by the same partner; attention to this feature may have supported task progress that this pair did not achieve. These earlier observations also revealed significant task features that appeared to remain un-noticed and certainly not acted upon by either pupil.

It was clear in observing these incidents that expression of awareness, however valid, was not sufficient for it to have impact on task progress. This raised questions for me which this study aims to pursue. In terms of the relations between pupils these opportunities to observe have shaped the current study in considering how the mixed attainment partnership can be mathematically productive without ‘fall[ing] prey to issues of status’ (Langer-Osuna, 2016, p108). In the current study I seek to explore the role played by the interaction between pupils in both stimulating awareness and influencing subsequent use of awarenesses. Building on the finding that a focus on noticing can be particularly beneficial to lower attaining pupils (Hohensee, 2016) I am interested in how mathematical awareness, can be productively influenced. Chapter 2, Section 2.4 sets out the precise aims and research questions that frame the study.

This study takes the form of a classroom-based intervention drawing on Education Design Research in its approach. The intervention aims to promote mathematical awareness for all pupils through a set of pedagogical adjustments. The study takes place in three Key Stage 2 mathematics classrooms in three different primary phase schools; data collection focuses on lower attaining pupils working with higher attaining partners. I employ the use of video recording to capture pupils’ activity and interaction as they work at their tasks. As is described in Chapter 3, the three class teachers were closely involved in the design of the intervention and through workshop discussion, in initial interpretation of some data extracts.

This study has the potential to make several contributions to the mathematics education field. Firstly, this research places lower attaining pupils centre stage and explores pedagogies that aim to enable them to gain from and contribute to
collaborative mathematical learning. It therefore aims to establish pedagogies that promote greater equity in mathematics learning. As such one potential contribution is in identifying practical changes that primary school teachers can make to promote more successful learning and higher attainment for this pupil group. Secondly, the relative paucity of research evidence focused on learning in mixed attainment pairs or groups has been noted recently (Taylor et al., 2017). Together with the identification of pupil noticing as a relatively under researched field (Lobato et al., 2013), this study has the potential to contribute to knowledge within these fields. Thirdly, my attention to the role of authority relations in mixed attainment learning offers the potential to contribute theoretical understanding that will both assist teachers in developing pedagogic practices and contribute to the research field in deepening understanding of learning processes in the social world of the classroom.

This thesis starts by presenting and examining the literature that has provided a foundation for the focus of the study (chapter 2). This is organised into three main sections, each of which addresses a component of the conceptual framework which focuses this study. Chapter 2 concludes with articulation of the aims and specific research questions of this study. Chapter 3 outlines the ontological approach underpinning the study and presents my research methodology and methods. This is followed by presentation and analysis of data in chapters 4 and 5, with discussion of practical and theoretical implications of the study outcomes presented in chapter 6. The final conclusions chapter presents contributions and recommendations arising from the study, considers its limitations and suggests avenues for further research.
Chapter 2: Literature review

This chapter examines literature drawn from the three key areas which comprise my conceptual framework and is accordingly split into three main sections. The first addresses mathematical learning for lower attaining pupils. A thorough examination of prevailing provision together with a discussion of how alternative approaches might better meet lower attainers’ mathematical learning needs enables me to establish a rationale for my focus on this pupil group. The second main section of this chapter addresses mathematical awareness. This is the focus of the planned intervention; discussion in this chapter positions awareness in relation to mathematical reasoning and explores both the construct of awareness and its relevance to mathematical learning. The third area of focus is informed by the critical realist ontology that underpins the research and which is set out in chapter 3. This ontology views social relations between individuals as central to the observable events in the open social world of the classroom. This section examines the authority relations that may emerge as pupils work together on their mathematical tasks and considers how such relations may contribute to the mathematical learning that takes place.

Each main section is split into several subsections which address topics within the main focus area. Each main section concludes by identifying implications of the preceding discussion for the current study. It is in these summary sections that the way in which the literature guides the development of the intervention and the analytic frameworks is articulated.

This chapter concludes with the articulation of the aims and research questions that arise from this analysis of literature.

2.1 Meeting the mathematical learning needs of lower attaining pupils

In this section I begin by defining the study population. I then move on to consider the predominant organisational arrangements for mathematics teaching and learning for lower attaining pupils in primary schools in England and review literature which evaluates the impact of these organisational arrangements on the learning of
this pupil group. Next, I discuss findings from studies in which alternatives to these arrangements have been used, noting that literature in this area is limited. Drawing on the positive outcomes of this limited range of studies, I briefly report underpinning pedagogy, beliefs and thinking that contribute to learning gains for lower attainers. In the summary of this section I present a case for conceptualising provision for lower attainers differently to the current norm.

2.1.1 Defining lower attainment

Low attainers in primary mathematics are a heterogeneous group of pupils (Houssart, 2004; Haylock, 1991). The term includes those who have identifiable learning disabilities or who face affective challenges, together with those whose low attainment stems from shortcomings in the way that the subject is taught (Haylock, 1991). My definition of low attainment aligns with that of authors who use the term low attainment to identify a group of pupils who attain less highly than their same age peers in relation to national expectations, but without identifying a cause or implying a pupil deficit model (Ghani and Kaur, 2012; H Barnes, 2005; Haylock, 1991).

In the UK, what counts as attainment below age related expectations and thus how ‘low’ or ‘lower’ attainment might be delineated has been complicated in recent years by changes in the national testing arrangements made in 2016 for key stage 2 (KS2) at the end of primary education (age 11) and in 2017 for GCSE examinations at the end of key stage 4 (KS4) at age 16. These changes, recognised as reflecting higher expectations (DfE, 2016b), have effectively redefined age related expectations and thus make direct comparison between pupils tested under different arrangements difficult. In identifying ‘lower attaining’ for the purposes of this study, data is drawn from pre-2016 test arrangements at KS2 and pre 2017 testing arrangements at GCSE.

Prior to summer 2016, expectations for primary school pupils’ mathematics achievement in England and Wales were couched in terms of numerical levels of attainment. The expected attainment at the end of KS2 (age 11) was level 4. The number of pupils not meeting this expected attainment level has fallen gradually since these indicators were introduced in 1995, to its lowest figure of 13% in 2015 (DfE, 2015). The group of pupils achieving below age expected standards across the
time period 1995-2015 includes a relatively static 3% (DfE, 2015) of all pupils for whom below test level functioning results in them not being entered for the test. Particular conditions or special educational needs frequently account for the significantly weaker attainment of this 3% of pupils. This small group are not the focus of this research. Rather my focus is on the approximately 10 - 15% of pupils who engage in age related mathematics learning, but who persistently struggle to keep pace with their peers. Whilst these pupils are entered for KS2 tests and are able to access many of the questions, they either do not achieve age related expectations or achieve them only at the threshold level following significant pre-test intervention and revision. Official statistics show that these pupils are highly likely to continue to struggle in the subsequent age phase of education; in 2016 just 21% of pupils who had attained level 3 in their 2011 end of KS2 tests went on to achieve the national expectation of grade C or above in their GCSE mathematics, whereas 69% of those achieving level 4 went on to achieve GCSE grade C or above in the same year (DfE, 2016a).

In this study, lower attainment is thus defined relative to age related expectations and as a result, relative to their peers in the mixed attainment pairs in which they worked. They represent a significant minority of our primary aged pupils; I would argue that their achievement trajectory provides a strong case for examining their current primary mathematics provision and considering whether and how learning can be more effective.

In the sections following, studies which focus on pupils whose low attainment has been linked to specific special educational needs have not been included in this discussion for reasons given above. However, the attainment-based definition of low attainment under which the majority of research studies operates means that the pupil population is necessarily diverse and may include pupils with some elements of identified need that are not specified in the research reports. Where researchers have provided pupil characteristics these are noted in the text below.
2.1.2 Provision based on notions of ‘ability’

The organisation of mathematics learning in both primary and secondary schools is frequently found to have features which contribute towards limiting progress and attainment for lower attaining pupils. Two common organisational arrangements are, firstly the re-organising of pupils across a year group to create a number of ‘ability’ ranked mathematics sets, and secondly the teaching of pupils across the attainment spectrum in ‘mixed ability’ classes. The term ‘ability’ is presented here in inverted commas to highlight its problematic nature; Francis et al (2017) note that the term conflates measures of a pupil’s current attainment with a presumed judgement of their innate capacity. Whilst the term ‘attainment’ is preferred and used within this thesis, Houssart (2004) notes that this term still carries a presumption that pupils have been offered appropriate opportunities to attain. Anthony and Hunter (2017) suggest that practices associated with notions of ability serve to label pupils, for example as ‘additive thinkers’, rather than describing their current learning behaviours. Such concerns lead to an assertion that ‘ability’ labels, with the presumptions that underpin them, and the practices that arise from them, run counter to ideas of equal opportunities (Dixon et al., 2002; Gillborn and Youdell, 1999). Indeed a recently published study describes teachers as being caught between ‘contradictory discourses’ of ability and inclusion in managing provision for lower attainers (Alderton and Gifford, 2018, p64). Yet, despite the lack of clear definitions or evaluation of their validity, discourses of ‘ability’ are pervasive in primary education (Marks, 2014; 2013). Indeed, the linking of the use of ‘ability grouping’ to the aim of raising educational standards has ensured that ability discourses have continued to shape practices and thinking in schools (Marks, 2011; Hallam et al., 2004a; 2004b).

In relation to the use of ranked mathematics sets, research evidence presents a consistent finding that such arrangements are detrimental to learning and progress of those in lower sets (Francis et al., 2017; Marks, 2014; Macqueen, 2013; Wilkinson and Penney, 2014; Zevenbergen, 2005; Kutnick et al., 2006). Authors identify a range of contributory factors including misallocation to sets with limited opportunities to move between them, lower teacher expectation and skill, a restricted curriculum,
disaffection, lack of motivation and the development of negative self-images relating to mathematics learning (Francis et al., 2017; Marks, 2014; Macqueen, 2013; Dunne et al., 2011; Solomon, 2007; Kutnick et al., 2006; Zevenbergen, 2005; Gillborn and Youdell, 1999). It should also be noted however, that whilst setting appears to particularly disadvantage those in lower sets, the development of negative self-images in relation to mathematics learning has also been found in pupils in higher sets (Mulkey et al., 2005; Boaler, 2005; Boaler et al., 2000). Despite this, research evidence suggests that the incidence of setting in primary schools has increased over the last 15 years. In 2003, in a sample of some 800 primary phase schools, the incidence of within year group setting was found to rise throughout key stage 2 to a peak of 24% of year six classes (Hallam et al., 2003). More recent research (Dunne et al., 2011) found that 44% schools albeit from a smaller sample (302 schools) operated a setting practice in year six; a separate study identified the incidence of setting by year 2 to be 34% of a sample of 727 schools (Hallam and Parsons, 2013).

Where pupils are organised into ‘ability’ groups in ‘mixed ability’ classes research indicates that for lower attaining pupils, some of the same issues can arise. For example MacIntyre and Ireson (2002) found that pupils in ‘low ability’ groups in mixed classes developed similar low self-images in relation to mathematics as were found in lower mathematics sets. In relation to task structure and opportunities for interaction Kutnick et al (2006) reporting case studies from 12 primary schools, found that ability group seating within mixed classes had implications for the type of task in which pupils engaged. Lower attaining pupils were offered the most limited range of pupil grouping and task structures, with individual work within small ‘ability group’ seating predominating. This resulted in the most limited opportunities for dialogue and collaboration. Kutnick et al (2006) also noted that where dialogue between low attaining pupils occurred, the placement of these pupils together meant that they were less likely to share understandings that could enhance learning for the group. Lastly the finding that pupils in lower sets experience a restricted mathematical diet is also evidenced where within class ‘ability’ grouping structures operate. This typically manifests in lower attaining pupils completing tasks that are repetitive, and which focus heavily on securing number facts and processes (Boaler et al., 2000;
MacIntyre and Ireson, 2002). Dunne et al (2011) concur, finding that learning for ‘low ability’ groups is at a slower pace, with reduced challenge, greater scaffolding and with a predominance of activities designed to reinforce the use of skills and methods. Such tasks typically offer few opportunities for higher order mathematical thinking (Macqueen, 2013) such as independent thinking and reasoning.

In seeking to explain these findings, Marks (2014; 2011) suggests that this typical paucity of opportunities for ‘lower ability’ pupils may be connected with teachers’ beliefs in relation to pupils’ mathematical needs and low teacher expectations of these pupils’ capacity to engage mathematically both with each other and with mathematical ideas. Differing opportunities for pupils to engage mathematically has also been seen to be embedded in teacher – pupil dialogue (Black, 2011; 2004) and to the disadvantage of lower attainers. This further contributes to the assertion that ‘ability group thinking’ is relatively impervious to change, (Gillborn and Youdell, 1999) and that it influences expectation, participation and interaction on many levels within the classroom (Black, 2011; 2004; Marks, 2013).

Beyond the individual school, Francis et al (2017) conclude that setting and grouping practices appeal to widespread views of natural order and social hierarchy and to prevailing middle class aspirations. Within the school, arguably, the placement of pupils into ability groups constructs expectations of their performance and participation in mathematics learning. These expectations are then reinforced through the provision of tasks that constrain types and levels of engagement and independence (Boaler et al., 2000; Macqueen, 2013; Mazenod et al., 2018). Thus, as Zevenbegen (2005, p607) notes, school practices reify the differences between pupils; they are ‘seen to be natural rather than being a consequence of the practices themselves’.

With the considerable weight of evidence providing a strong caution against the use of ability grouping, it is perhaps surprising that the use of these structures remains so prevalent. Here, research reveals that factors underpinning the use of ability groups include views that such structures support effective targeting of levels of difficulty of work, enable provision of support in the form of teaching assistants for
‘low ability’ groups, render learning less intimidating for pupils, and support teacher confidence (Anthony and Hunter, 2017; Hallam et al., 2003). Research also indicates that there may be a lack of clarity amongst teachers in relation to how different grouping structures can support pupil learning (Anthony et al., 2016). Even where teachers express dissatisfaction with their school’s current use of ability grouping, the use of alternative grouping structures is seen as risky, needing a staged introduction, and significant teacher development and support (Anthony and Hunter, 2017; Taylor et al., 2017; Daniel, 2007).

Literature in this section paints a troubling picture of a prevailing situation that actively limits learning for lower attaining pupils, but with little apparent appetite for or confidence in making changes to better support these pupils. In the following section I examine the extent to which lower attaining pupils can engage in the higher level mathematical thinking that might render them able to work alongside their higher attaining peers.

2.1.3 Lower attaining pupils can, and do, think mathematically

The relatively limited range of literature that addresses the potential of lower attaining pupils to engage in higher order mathematical thinking is perhaps not unconnected to the prevailing task structures and working pace for low attaining pupils outlined in the previous section. However, where research evidence exists it is consistent in recognising the potential for lower attaining pupils to engage in mathematical thinking.

For example, Dickinson et al (2010), found that approaches incorporating investigative, problem solving tasks based in a realistic context resulted in strong learning gains for lower attaining pupils. These gains were revealed in later problem solving based tests in which the focus pupils were more successful than similar attainment peers who had not engaged in the investigative approach. Focus pupils were seen to more frequently employ sense making strategies such as drawing and annotating diagrams as part of their approach to solving problems. The authors stress the importance of the use of models and images in supporting pupils to develop sense-making approaches. This study focused on pupils who had been in the lowest
40% of performance in their recent KS2 tests at the end of primary schools. The year in which data was collected is not specified in the report, however pupil attainment of age related expectations at the end of KS2 has been greater than 70% since 2000 (DfE, 2015). Therefore this data set is highly likely to extend beyond those pupils who did not meet KS2 age related expectations and thus includes but extends beyond those on whom I will be focusing.

Also focused on secondary education, Watson and De Geest (2005) worked with teachers who developed pedagogies and activities that were underpinned by their common focus on pupils making choices, exercising freedom, engaging in challenge and taking responsibility. Tasks devised did not follow a single style, but shared commonality in the levels of mathematical thinking required. Pupils in this study were low attainers working with similar attainment peers; whilst low attaining these pupils had no specific special needs that would account for their low attainment. Over the course of the study, development in a range of aspects of pupils’ higher order mathematical thinking was noted: their facility to generate ideas, to make conjectures, to generalise and justify and to explore possibilities all showed improvement. Watson and De Geest (2005) identified a set of shared beliefs amongst this group of teachers that provide a welcome counter to the findings of Marks (2014; 2011) reported above. Unlike the low expectations reported by the latter, Watson and De Geest (2005) found that teachers in their study were committed to promoting higher level mathematical thinking and engagement in their lower attaining pupils. Beliefs in common were that all students could learn mathematics; that all students had a right to engage in genuine mathematical thinking and to learn in a way that developed their reasoning; and that the teacher’s role was to support learning without artificially narrowing the mathematical complexity of ideas.

Whilst the above studies have focused on secondary aged pupils, Houssart’s (2004; 2002; 2001) observational studies focus on primary aged pupils. Houssart notes that in situations where higher order mathematical thinking was not part of the set task, lower attaining pupils nevertheless made comments which reflected both a capacity to and an interest in engaging mathematically. Such comments, identified by Houssart (2004; 2001) as ‘whispers’ due both to the volume at which the comment
was made, and to some ambiguity as to the intended audience, were seen to address a range of types of mathematical thinking and reasoning. Houssart (2004; 2001) identifies three main types of ‘whisper’: those reflecting a mathematical noticing or discovery, such as a generality or an unexpected outcome; those reflecting an extension to an idea under discussion, for example a pattern continued or an idea further developed; and those reflecting disquiet at a statement made by another which contained a mathematical inaccuracy. Such comments, arising from the mathematical topics under discussion, but not always directly relating to the task in hand confirm that pupils identified as low attainers can display the mathematical thinking more often associated with pupils who are identified as more able mathematicians.

Thus, whilst both Dickinson et al (2010) and Watson and De Geest (2005) demonstrate that lower attainers can engage successfully in mathematical thinking, investigation and problem solving when specifically supported to do so, Houssart (2004; 2001) shows that such behaviours are part of how such pupils operate independently. This may go some way in explaining why approaches which harness these capacities have a high chance of supporting further mathematical development. This is in line with Watson’s recommendation that provision for lower attaining pupils should be founded on a proficiency, rather than a deficiency model (Watson, 2001a; 2001b). She reasons that building on the thinking skills that lower attainers demonstrate and using these to support exposure to new ideas is more profitable than an approach that focuses on gaps in their knowledge.

Whilst the above studies demonstrate that when working with similar attainment peers, lower attaining pupils can demonstrate aspects of mathematical thinking that are more commonly found in their higher attaining peers, the next section considers how such capacities are applied when lower attaining pupils work as part of a mixed attainment pair or group.
2.1.4 The potential of mixed attainment groupings

The inequality of opportunity and inequity of outcome for lower attaining pupils that appears to permeate provision based on notions of ability begs the question of how improvement for this group of pupils can be made without detriment to the learning and progress of other pupil groups. Since research in the preceding section has shown that lower attainers can think mathematically and respond to challenging tasks, it could be argued that there is no need for altering ability grouping structures, but only to improve provision for lower attainers within the prevailing ‘ability’ based organisation. Indeed, findings from Watson and De Geest (2005) show that teacher belief in the potential and entitlement of lower attainers, combined with developments to pedagogy to ensure the provision of appropriate challenge, constitutes one such route towards this improved provision. However, research literature also suggests that mixed attainment groupings can provide benefit for all pupils. It must be acknowledged that this evidence is somewhat limited, not least because of the resistance of schools and teachers to trialling alternative approaches (Taylor et al., 2017; Anthony and Hunter, 2017). This section considers such literature but also draws from a significant body of work relating to the development of equitable provision in which heterogeneous grouping reflects a wider cultural and socio-economic diversity in the pupil groupings. In such literature attainment differences are included in consideration but are not the prime focus for analysis; the findings of these studies nevertheless remain highly relevant to the debate.

At the outset it is important to note that removing the overt ‘ability’ based grouping practices does not remove the ‘ability’ based thinking that underpins them (Marks, 2016; Dixon et al., 2002). Indeed Dixon et al (2002) note that the use of and reference to ‘mixed ability groups’ requires ability based thinking in order to define such groups. Thus it can be argued that whilst appearing to be more grounded in an equitable approach, such practice still reflects a deterministic view of educational potential. In response, Dixon et al (2002, p9) promote a ‘transformability-based’ approach that views pupils’ futures as being created in and by their experiences at school and bases pedagogies on a belief of universal entitlement.
Where mixed pupil groupings have become established, one notable gain is in the development of mathematical behaviours. For example, teachers of primary aged pupils working in mixed attainment groups report the development of greater pupil ownership of their mathematics learning, of pupils learning to listen to and learn from each other, and learning to value different approaches (Anthony and Hunter, 2017). Anthony et al (2016) also noted how views of what constituted valuable mathematical activity altered with a change to mixed attainment groupings, from a focus on knowledge to that of developing mathematical practices. More pupils believed they could ‘do maths’ and across teachers and pupils a mindset that all could learn and improve began to develop.

The available attainment and progress data indicates that the use of mixed attainment (or heterogeneous) grouping results in positive outcomes for all pupils. For example, Leonard (2001) compared the progress, over the course of an academic year, of homogenous and heterogeneous triads of pupils in naturalistic class based mathematics learning in a US elementary school. Based on pre and post-test functional mathematics assessments she found that all those working in mixed groups fared better than those working in homogeneous groups, with the effect being most pronounced for previously middle and lower attainers.

High teacher expectations and exposure to challenging mathematical ideas underpinned the approach developed in Watson and De Geest’s (2005) study reported in the previous section in which low attainers worked with others of similar low attainment. Findings from studies involving mixed groups identify the same features of teacher belief and approach as significant in underpinning learning gains. For example, focusing on heterogeneous classes in US grades 6-8 (equivalent to UK years 7-9), Burris et al (2006) employed the principles of using an accelerated curriculum for all students. The authors found that this approach led to 100% success in the core mathematics standard studied. Furthermore, rather than depress achievement of those who would otherwise have been in top sets, this approach led to greater student participation and success in higher level mathematics study in later grades. This further mathematical engagement included more students at initial lower achievement levels opting to continue to study higher level mathematics.
classes than had previously been the case. Linchevski and Kutscher (1998), also researching in secondary age classrooms, similarly conclude that exposure to a wider range of topics at a higher level is significant in promoting greater progress for lower and middle attainers. Lastly, Boaler’s comparison studies (2010; 2008) also find higher levels of mathematical achievement and future mathematical engagement in pupils who had experienced mixed grouping practices. Here, the high expectations that teachers had of all pupils and the opportunity to learn to value the skills and aptitudes that all pupils brought to the tasks were cited as particularly significant (Boaler, 2010; 2008).

The necessity to provide a task which all pupils within a mixed attainment group can access leads frequently to the use of ‘open structured’ tasks (Boaler and Staples, 2008; Boaler, 2016). This task type presents investigative or problem solving challenges characterised by opportunities for multiple ways of working, different pathways of exploration, an accessible starting point and intellectual challenge (Boaler and Staples, 2008; Boaler, 2016). Their goals and structure also encourage and afford pupils opportunities to engage in mathematical practices of reasoning, justifying, explaining, representing, generalising and developing and using mathematical language (Boaler, 2016). These opportunities tend to lead to greater levels of engagement from all group members and promote greater collaboration and more equitable participation (Boaler, 2016; Boaler and Sengupta-Irving, 2016; Boaler and Staples, 2008; Engle and Conant, 2002; Alexander et al., 2009; Turner et al., 2011). Put simply, in a task where no answer or route is immediately obvious or prescribed, there is more likely to be a discussion (DeJarnette and Gonzalez, 2015). These tasks promote reciprocal help seeking and place it within a context where, echoing the findings of Anthony et al (2016) above, work is focused on developing mathematical practices, not merely on finding solutions (Boaler, 2016; Boaler and Staples, 2008; Cohen and Latan, 1995).

Whilst this body of research points strongly to a connection between task structure and greater engagement, it is important to establish how such features may contribute to learning for all pupils in the group. It is of course not guaranteed that such tasks will promote either equitable participation (Turner et al., 2011), or
learning. Further, as noted above, the body of literature focused on equitable engagement has not focused analysis specifically on learning gains for lower attainers. However, for such gains for the lower attainer to ensue, Fawcett and Garton (2005) stress that active participation of the lower attainer throughout the task is important. This requirement may be difficult to realise given that a history of limited participation may impact negatively on the likelihood that a pupil takes up opportunities to participate (Foote and Lambert, 2011). Further, lower attainers may not recognise the role that participation has for their own learning (Black and Varley, 2008), and may not view their contributions as likely to be of value (Black and Varley, 2008; Cobb et al., 2009; Bishop, 2012).

Where learning gains for all pupils in mixed groups are identified, whilst some authors assert that increased levels of interaction are a predictor of learning (Cohen and Latan, 1995), most argue that it is the nature of the interaction that is significant, not just the quantity (Grootenboer and Jorgensen, 2009; Gresalfi et al., 2009; Imm and Stylianou, 2012; Esmonde, 2009a; Boaler and Staples, 2008; Wood et al., 2006; Mercer and Sams, 2006). Focusing specifically on analysis of attainment, and with primary age pupils, both Whitburn (2001) and Leonard (2001), report that both the amount and quality of interaction between higher and lower achievers is significant in supporting learning for the lower attainer. Fawcett and Garton (2005), also working with primary aged pupils concur, specifically noting that gains for lower achievers occurred because of their exposure to higher levels of reasoning when working with a higher achieving partner. Importantly they also emphasise the need for active participation, noting that in their study in situations where talk between mixed pairs was discouraged, there was a tendency for pupils to take turns and frequently to become disengaged from their partner’s activity. Conversely the requirement to explain meant that both pupils engaged in exploration and evaluation of the activity that was undertaken. Roberts (2016), working with pupils in year 7 in a literacy context, identifies benefits for both pupils through ‘reciprocal learning relationships’ (p50) in which questions posed by the lower attainer prompted the higher attainer to clarify and expand on their thinking. She reports that the ideas emerging from such exchanges benefitted both pupils. Webb et al (2014) build on
this further in noting that whilst being involved in explaining ideas is positively related to achievement, higher levels of engagement such as having ideas challenged, engaging in justification, questioning and refining have a greater impact on achievement.

2.1.5 Summary and implications for the current study

Literature presented in the preceding sections confirms firstly that lower attaining pupils have the capacity to engage in the kind of mathematical reasoning that is more usually considered to be a feature of the activity of their higher attaining peers (Watson and De Geest, 2005; Houssart, 2004). Evidence that the use of open structured tasks provides opportunities for the development of mathematical practices and engagement in mathematical reasoning (Boaler and Staples, 2008; Boaler, 2016), provides a firm grounding for the use of such tasks to support mathematics learning in this study; these tasks are further discussed in subsequent sections.

Literature has also shown that mixed groupings have the potential to support lower attainers (Anthony and Hunter, 2017; Boaler, 2010; 2008; Leonard, 2001). Several factors emerge from research as significant in enabling these benefits, including exposure to increased mathematical challenge, to high expectations and to the more sophisticated mathematical thinking of their peers. Moreover, evidence that higher attainers do not suffer from this arrangement and indeed benefit from the requirement to justify and explain their thinking should allay some of the concerns raised by teaching professionals (Anthony and Hunter, 2017; Taylor et al., 2017; Daniel, 2007) in relation to making changes to existing grouping practices.

However, the literature examined in preceding sections leaves an important question unanswered. This question provides an additional rationale for the use of mixed attainment pairs and represents one contribution to the field that this study can make. In the preceding discussion, learning gains for the lower attainer in the context of mixed groups have been detailed as arising from their partnership with the higher attainer and through the affordances of the task; gains for the higher attainer have been reported as arising from the requirement to explain and justify their own
thinking (Roberts, 2016; Webb et al., 2014). Thus, the existing research appears to suggest that gains for both higher and lower attainer derive from what the higher attainer contributes. There is currently no evidence that indicates that lower attainers contribute independently to the mathematical understanding or progress in a task that a mixed group makes. This is despite research evidence reported above that indicates that lower attainers are also capable of the thinking and reasoning required to make valid contributions (Watson and De Geest, 2005; Houssart, 2004) and that they can respond to exposure to challenging mathematical ideas and tasks (Dickinson et al., 2010; Burris et al., 2006). Thus, this research seeks, through its focus on the mathematical contributions of the lower attainers, to identify and report the ways in which lower attainers contribute mathematically to the mixed attainment partnerships in which they participate. The extent to which lower attainers’ contributions promote mathematical progress in the task for both pupils is a focus for close analysis.

My focus on the mathematical contributions of lower attainers is thus in the context of mixed attainment pairings. It draws on a focus on promoting mathematical awareness to which discussion now turns.

2.2 Awareness and Reasoning in pupils’ mathematics learning

Mathematical reasoning is frequently identified to be fundamental and central to both the discipline of mathematics and to the learning of it in school (Ball and Bass, 2003; Lithner, 2000a; Yackel and Hanna, 2003; Yankelewitz et al., 2010; Mueller and Maher, 2009). Indeed Ball and Bass assert that it is not tenable to speak of mathematical understanding without ‘a serious emphasis on reasoning’ (2003, p28). Reasoning enables mathematical facts and ideas to become connected, and for mathematical understanding to consist of more than competence in recall and the following of procedure (Ball and Bass, 2003). Skemp (1979) asserts that this connection and articulation of symbol and ideas evidences logical understanding; a different and higher quality of mathematical understanding. Such assertions of the centrality of reasoning to understanding are unchallenged in mathematics education research, and significant authority is added to these claims by the findings of Nunes
et al (2007; 2012). Drawing on data from assessments of reasoning and arithmetic competence of over 2,500 pupils at different stages of their education, Nunes and her colleagues (2007; 2012) established that pupils’ strength in reasoning is a stronger predictor of their later mathematical achievement than is the ability to carry out arithmetic processes.

The identification of reasoning as one of three high level aims of the National Curriculum in England (DfE, 2013), alongside fluency and problem solving, arguably reflects recognition by UK policy makers of the importance of reasoning in pupils’ mathematical development. However, subsequent scant reference to reasoning in the curriculum’s programme of study statements could be seen as a diminishing of its status relative to that of fluency, which these statements more overtly reflect. Nevertheless, more recent pedagogical guidance which emphasises the role of reasoning in the development of both fluency and problem solving skills (Askew et al., 2015), together with an increased focus on reasoning in end of primary phase national testing, re-emphasises its position as a central component of pupils’ mathematical learning.

In this section I begin by considering key factors contributing to the development of pupils’ mathematical reasoning. I then use these factors to position mathematical reasoning in relation to mathematical awareness, and to articulate the significance of the latter in developing the former. I then move on to focus consideration specifically on the construct of mathematical awareness and tease out its connections with related constructs of noticing and attention. Next, I consider the particular relevance of the construct of awareness in pupils’ mathematics learning and discuss how pupils’ mathematical awareness may be stimulated. Following this I combine literature outcomes from this section with those from section 2.1 in discussing how a focus on awareness might particularly benefit lower attainers in mathematics. Lastly, I consider implications arising from the preceding discussion for the inference and analysis of awareness in the current study and set out the definition of awareness that guides my analysis.
2.2.1 The development of pupils’ mathematical reasoning

Reasoning must have a base upon which it is constructed and several authors identify the importance of conceptual knowledge in providing such a base (Kim and Kasmer, 2007; Francisco and Maher, 2005). Considering the development of mathematical reasoning amongst primary aged pupils in the context of class mathematics activity, Ball and Bass (2003, p31) define a ‘base of public knowledge’. This, they argue is knowledge shared by a particular class or group and is knowledge which can be invoked without the need for additional verification. It includes facts, ideas, procedures, terms and methods of investigation and proof. This base of knowledge is particular to, and dependent upon, the collective experiences and history of the class or group; Ball and Bass (2003, p32) note that its extent, depth and security can both ‘constrain and enable’ the formulation of argument, a finding also noted by Wistedt and Brattstrom (2005).

In addition to the affordances and constraints exerted by the depth and security of the knowledge base, the degree to which an appropriate selection from this base, of ideas, facts or procedures is made will also impact on the quality of reasoning. Here, Bergqvist and Lithner (2012, p254) establish the idea of ‘anchoring’. This refers to the selection and use of relevant mathematical properties of the object or concept that is the focus of reasoning. Bergqvist and Lithner (2012) identify both surface and intrinsic properties of concepts, and, whilst acknowledging that what counts as a surface or intrinsic property is dependent on context, stress that appropriate selection is fundamental to the validity of the ensuing reasoning. These authors note how problems arise when students attempt to anchor reasoning in a surface rather than an intrinsic property. Similar difficulties are also noted in other studies (Francisco and Maher, 2005; Wistedt and Brattstrom, 2005).

Lithner (2000b; 2000a) further identifies a concern that pupils rely too frequently on what might be termed ‘experiential reasoning’; that which is based on a previously used procedure or behaviour rather than being rooted in the properties of the mathematical situation (Lithner, 2000b; 2000a). This might be seen to call into question the inclusion, by Ball and Bass (2003), of ‘procedures’ in the base of public
knowledge they articulate. However, Lithner (2000a) refers to the use of a procedure or action because it has previously been used, based on a presumed congruence of contexts; this differs from the invoking of procedure or method of enquiry to justify the validity of an outcome. Thus it is not the use of the procedure that is the issue for Lithner (2000b; 2000a), but the reason for its use. It is this to which his term ‘experiential reasoning’ refers. Lithner (2000b; 2000a), instead argues that students should initially work with what he describes as ‘plausible reasoning’; that which is founded on mathematical properties of the idea in question and steers towards what is ‘probably true without necessarily being complete or correct’ (Lithner, 2000b, p167).

The use of ‘plausible reasoning’ resonates with Mason et al.’s (2010) identification of a cycle of generating, testing and gradual refining of conjectures initially based on what appears to be true but which has not yet been convincingly demonstrated to be so. However here Lithner (2000a) also cautions that students frequently do not recognise the importance of distrusting their own thinking and following this through with considering alternatives (Lithner, 2000a). This latter is also noted by Mercier (2011) in his discussion of ‘confirmation bias’; the seeking of examples that fit a conjecture rather than seeking those that do not.

The pathway to constructing reasoning is thus not without its challenges and complexities. These notwithstanding, primary aged pupils have been shown to be able to develop and articulate mathematically sound and convincing arguments drawing on established forms of direct and indirect reasoning; many examples are present in literature (Mueller, 2009; Mueller et al., 2012; Yankelewitz et al., 2010; Ball and Bass, 2003). However, the prime focus of this study is not on the nature or quality of the reasoning that pupils are able to construct, but on the awareness and appropriate use of a mathematical knowledge base upon which reasoning can be founded. The nature of mathematical awareness and its relationship with reasoning is discussed in the following sections.
2.2.2 Positioning mathematical awareness in relation to mathematical reasoning

Mathematical reasoning is widely agreed to be a process actively engaged in (Mueller, 2009; Mason et al., 2010; Mercier, 2011; Mueller and Maher, 2009). Its argumentative purpose and use of a knowledge base is exemplified in the definition of reasoning as ‘the revisiting and reconstruction of previous knowledge in order to build new arguments’ (Mueller and Maher, 2009, p7). However, the selection from this knowledge base (Ball and Bass, 2003); the anchoring of reasoning in a relevant property (Bergqvist and Lithner, 2012), and the generation of a conjecture (Mason et al., 2010), identified as significant in the preceding section, all require an ability to discern features and actively choose to build on them in constructing an argument. This is best reflected by Francisco and Maher (2005, p364) who argue that reasoning is ‘associated with the ability to discern and articulate’ the relationships that exist within mathematics.

The acknowledgement of the significance of awareness, or noticing of, features on which to base reasoning pervades reasoning literature; often this is found in the description of events rather than in the definition of reasoning an author constructs. The terms used vary between authors, but common to all is the involvement of perceptual discrimination as a foundation for reasoning. So, for example, Mercier (2011) identifies perception as the basis for inference; Lithner (2008; 2000a) reports how students identified similarity, noticed anomalies, perceived properties. Similarly, Nunes et al (2009) describe processes of discerning patterns, detecting structures, becoming aware of relations. Lastly, Reid (2002) describes how pupils noticed number size, observed features and were able to ‘see’ a feature. It is noteworthy that these authors also use these terms in describing their own processes of constructing arguments about their findings.

Together the frequent use of these terms to describe the perceptual discrimination that leads to the construction of an argument raises my own interest in the development of this early stage of reasoning. However, the fact that definitions of reasoning frequently do not make explicit mention of this initial noticing or awareness may reflect a view that this perception is a precursor to, rather than an
intrinsic element of, the reasoning which builds upon it. The close relationship between awareness, or noticing, and reasoning is reflected in Mason’s (2002, p7) assertion that ‘what you do not notice, you cannot act upon’. Whilst drawn from a different educational context this comment highlights that awareness of a feature is necessary if this feature is to be used. This does not imply however, a direct correlation between features noticed and features used in reasoning. Indeed Lobato et al (2013) note that in their study not every focus of pupil attention was selected as a basis for reasoning. I thus position awareness, or noticing, as a necessary but not a sufficient condition for reasoning.

This distinction notwithstanding Lobato et al (2013) argue that highlighting ‘noticing’ for specific focus, rather than treating it as part of the reasoning process, constitutes a valuable research approach. They suggest that this specific focus supports explanations of the quality of, and limitations in, pupils’ reasoning. Examining the particular features on which reasoning has been based and through this identifying what pupils have and have not noticed and selected for use may thus support understanding of the selection of surface or intrinsic properties (Bergqvist and Lithner, 2012). It may also support understanding of why particular features of a knowledge base (Kim and Kasmer, 2007; Francisco and Maher, 2005; Ball and Bass, 2003) are, or are not, drawn upon.

Lobato et al’s (2013, p846) comment that pupil noticing is currently an ‘under researched construct’ provides further momentum for its centrality in the current study and it is to specific consideration of this construct that discussion now turns.

2.2.3 Awareness: Exploring related constructs

Gattegno argues that knowledge ‘comes to life through awareness’ (Gattegno, 1987, p51). His assertion that awareness is at the centre of the learning process leads Young and Messum (2011) to assert that this means that an awareness, the event that occurs between not-knowing and knowing, can be considered as the ‘conceptual unit of learning’ (p83). Indeed, awareness of relationships and actions is at the root of the development of the discipline of mathematics itself (Gattegno, 1987; 1988; Mason, 2008a). Gattegno further considers that the human mind is well equipped to engage
in this discipline, here he draws on natural human tendencies to engage in activities such as identifying similarity and difference, stressing some features and ignoring others, and using imagery. He argues that these natural powers enable pupils to acquire their mother tongue without formal tuition and are the same learning powers that are required in learning mathematics (Gattegno, 1988).

Hewitt (2009b; 2001) notes that Gattegno does not specifically define awareness; Mason however suggests that Gattegno’s intent is that awareness in humans refers to our sensitivity in detecting stimuli and responding to these stimuli through action (2008a). Thus, awareness guides and informs actions and decisions (Mason, 2008a; Hewitt, 2001). The extent to which one is aware of the connection between awareness and action is significant; many actions or functionings, whilst triggered by an awareness, may operate at the subconscious level. Building on Gattegno’s work, Powell defines awareness in the context of mathematics learning as ‘a particularity that the perceiving mind preserves and that avails itself for future use’ (1998, p37). This definition stresses awareness both of the feature, or particularity, and of its potential relevance in future experience. As such this definition finds resonance in Mason’s argument that participation in the discipline of mathematics requires explicit awareness of the actions and functionings triggered by an awareness; this second order awareness means that these functionings can be studied, evaluated and more deliberately selected (Mason, 2008a). To clarify this further, Gattegno (1987, p40) states that:

Knowing is the awareness that one is aware of something, and according to whether we stress the something or the awareness, we progress in the subject or in the education of our awareness.

So, for example, awareness of a relationship between two numbers might trigger an action that involves searching for similar relationships in other sets of numbers. Whether one then focuses on the numerical relationship that emerges or reflects on the awareness of mathematical patterns and relationships, will lead to different learning.

Awareness and attention are closely linked (Hewitt, 2001; Mason, 2008a; 2008b). Hewitt (2001) notes a reciprocal relationship, with awareness influencing the
directing of attention, and the focus of attention influencing that of which one becomes aware. Mason (2011) explains that something can be attended to in a variety of ways: we may firstly ‘hold wholes’, that is we may look at something in a general way without discerning any particular features, a type of relatively unfocused gazing which may lead to a more specific attention. We may also ‘discern details’, by which he means we separate elements, discriminate one part from another. Mason also suggests that attention may be in the form of ‘recognising relationships’; identifying similarity, difference, or making relative comparisons. Lastly we may ‘perceive properties’, or identify particular relationships as instances of more general properties (Mason, 2011). Since, as Mason (2008b) argues, movements of attention can provide clues as to the nature of underlying awarenesses, these different ways of attending thus not only describe a different focus, but enable and reflect awareness of different features.

Mason relates the constructs of noticing and attention by asserting that noticing is a ‘movement or shift of attention’ (2011, p45). Mason suggests that whilst a noticing may arise spontaneously, it frequently requires an intentional shift of attention for one to become consciously aware of something. He notes that attention may be fleeting or sustained, conscious or at least partially sub-conscious, but that to notice something marks it as being attended to (Mason, 2011). In a definition that extends and broadens Mason’s conception of noticing, Lobato et al (2013, p809) define noticing as ‘selecting, interpreting and working with particular mathematical features or regularities when multiple sources of information compete for one’s attention’. This definition highlights the involvement of intent with the term ‘selection’ and in this respect bears similarities to Mason. However its implication of prolonged attention to and use of particular features or ideas means that the timescale and scope of Lobato et al’s (2013) definition is very different to the one described by Mason (2011) above (P. Messum 2017, personal email communication, 18th July).

An alternative analysis is provided by Noble et al (2004) who present a continuum of awareness represented at the one end by ‘seeing as’ and at the other by ‘not seeing a whole’. ‘Seeing as’ reflects a capacity to see something flexibly, in different ways and to switch between different ways of seeing. ‘Not seeing a whole’, arising either
from inexperience or unhelpful focusing, reflects being able only to see parts of an image or object but not being able to interpret the whole. In between the two extremes lies ‘recognising-in’ where a familiar aspect or feature within an unfamiliar whole enables the whole to be rendered less alien. This ‘recognising-in’ paves the way for more holistic and flexible seeing. Noble et al (2004) describe a process of ‘aspect dawning’ as pupils experience a change in the way in which they are able to see a feature. Their identification of ‘aspect dawning’ bears some similarities with Mason’s analysis whilst also bearing some differences: Mason’s (2011) shifts of attention may both reflect and enable awareness of different features and thus arguably promote the flexible, holistic viewing embodied in Noble et al’s (2004) ‘seeing as’. Noble et al’s (2004) ‘recognising-in’ has some resonance with Mason’s (2011) ‘discerning details’ and ‘recognising relationships’ in the discerning of parts within a whole and the use of these to make sense of the whole. However Noble et al’s (2004) interpretation of ‘seeing-as’ is not the same as the holistic gazing reflected in Mason’s (2011) ‘holding wholes’.

The literature above presents a consistency of view that constructs of awareness, attention and noticing are closely connected. Thus, to draw productively on these ideas for the purposes of the current study, three key issues arise. Firstly, that awareness arises from our natural tendencies to ‘stress and ignore’ (Gattegno, 2010; 1988); secondly that the close relationship between awareness and attention means that shifts in our attention (Mason, 2011), whether intentional or not, can provoke awareness of new features. Lastly, whilst differences persist in the way in which the constructs of awareness, attention and noticing are defined and subdivided, literature is consistent in identifying that there are different ways of attending, which enable focus to shift between part and whole, and where greater awareness enables more flexibility in the way in which we contemplate and examine mathematical ideas. The next section pursues these ideas further in considering the extent to which they can be productively harnessed in mathematics learning.
2.2.4 Awareness and mathematics learning

Gattegno views the aim of mathematics learning as to develop ways of seeing that enable learners to become aware of mathematical relationships. He considers these relations to be the defining feature of mathematics arguing that they ‘can be perceived as easily as objects’ (1987, p16). In support of this aim, Gattegno (2010; 1988) proposes an approach to teaching that deliberately seeks to stimulate awareness of a range of relations such as complementarity and equivalence as well as awareness of processes and objects.

This assertion of the significance of both awareness and of the role of relationships finds echoes in the later work of Mulligan and Mitchelmore (2009). These authors argue that mathematical relations give rise to structures and patterns and that ‘virtually all mathematics is based on pattern and structure’ (Mulligan and Mitchelmore, 2009, p33). Their research evaluated pupils’ awareness of structure and correlated this with mathematical attainment; findings demonstrated that each pupil’s level of structural awareness was relatively stable across the range of tasks and that higher levels of structural awareness were linked with higher levels of mathematical achievement. In relation to tasks addressing number, measurement and space, the authors identified four different stages of the development of structural awareness, ranging from a pre-structural stage in which no evidence of structural awareness was evident, through an emergent phase in which some disconnected elements of structure were apparent, to a partial structural stage in which elements of structure are present but incomplete, and lastly to a structural stage in which awareness of structure is evident and complete. Mulligan and Mitchelmore (2009) do not draw specifically on Gattegno’s work. However, their reference to the importance of detecting sameness and difference and seeking pattern points towards a similarity with this earlier work. Whilst not defining awareness Mulligan and Mitchelmore identify both a cognitive and a metacognitive aspect to it; they argue that knowledge of structure together with a tendency to look for pattern are reflected in how pupils perceive and respond to mathematical stimuli. Arising from this, the authors recommend greater focus on developing pupils’ awareness of structure. They conjecture that higher levels of structural awareness
may enable higher achievement because pupils may be able to apply their structural awareness successfully in understanding number relations and properties (Mulligan and Mitchelmore, 2009).

In proposing a model of learning processes, Gattegno (1987) positions awareness at centre stage. He articulates a model that begins with the awareness that there is something to be learned. Activity begins with a hesitant trial and error approach and moves towards more focused and deliberate trials in which awarenesses arising from trials inform refinements and adaptations to the approach taken. As learning progresses, awareness is increasingly less focused on the actions needed, and can instead be focused on the limits that may exist, or on what can be changed or manipulated, and on what is and is not possible (Gattegno, 1987; Young and Messum, 2011). This resonates strongly with the ‘manipulating - getting a sense of - articulating’ framework later described by Mason and Johnston-Wilder (2006) which incorporates cycles of increasingly fluent and confident activity as learners become accustomed to, make sense of, and articulate the mathematical relationships that exist in the context in question. Both models illustrate that learning requires a range of different types of awarenesses. For example, mathematical learning includes awareness of opportunity (e.g. to learn something), awareness of relevance (e.g. of a process or mathematical feature), awareness of outcome (a property, relationship), awareness of an implication (a constraint, structure, possibility, logical inference).

Both models also reflect constructivist approaches to learning mathematics which view learners as actively building their mathematical understanding through carefully structured direct experiences and opportunities to explore, refine and reflect. Indeed Hohensee (2016) draws specifically on the constructivist perspective in suggesting that what is attended to, and thus closely connected to what is noticed, plays a role in knowledge construction through contributing to processes of assimilation, accommodation and reflective abstraction. Hohensee (2016) notes that this can explain instances of both productive and less productive noticing. In the case of less productive noticing he cites the assimilation of a feature into an existing schema that is associated with an irrelevant aspect of the mathematics in question. Arguably such assimilation might then lead to the use of surface features on which
to base reasoning (Bergqvist and Lithner, 2012). Hohensee (2016) proposes that instances in which an unexpected outcome is noticed can result in accommodations made to an existing schema that constitute productive development of knowledge or understanding. The productive use of surprise is also noted by Hewitt (2009a) below. Hohensee (2016) proposes that reflective abstraction, that is to say, focusing on the nature of the awareness rather than on the outcome of it, can support transfer to novel situations. This latter resonates with Mason’s (2008a) identification of second order awareness, noted in 2.2.3. It is also supported by separate findings that noticing has ramifications beyond pupils’ current activity in providing a ‘conceptual rootstock’, influencing the nature of what pupils look for in terms of mathematical relations and thus how they later apply understanding in a novel transfer situation (Lobato et al., 2012, p476).

Since learning requires the development and use of awarenesses, it follows that Gattegno views that the teacher’s role is to structure learning in order that these awarenesses can be developed, a view endorsed by Powell (1998). Indeed Gattegno’s often quoted statement that ‘only awareness is educable’ (1987, p108) suggests that this encompasses all that a teacher can attempt to achieve. Building on Gattegno’s ideas, Hewitt’s (2001) call for the education of awareness makes the argument that awareness awakens the mathematician within the pupil, enabling independent mathematical decision making and action that goes beyond reproducing what has been memorised. Whilst awareness cannot be bestowed, only promoted, teacher pedagogy can, he argues, productively focus pupil attention (Hewitt, 2001). Indeed, such support is required to enable pupils to learn to look in discipline specific ways (Noble et al., 2004).

The finding that ‘what students notice mathematically has consequences for their subsequent reasoning’ (Lobato et al., 2013, p844) more firmly establishes the importance of pedagogy that aims to productively influence pupil awareness. Noting that little research currently addresses this topic (Lobato et al., 2013), findings from a small number of studies nevertheless confirm the assertion that encouragement to notice and use particular mathematical ideas increases pupils’ sensitivity to them. For example, Lobato et al (2013) report greater independent awareness and use of
multiplicative relationships where these had been specifically stressed through aspects of teacher pedagogy such as gesture, highlighting through language and the use of board annotations. Similar findings are reported by Voutsina and Ismail (2011) working with low attaining pupils in a different context. Their focus on recognising, using and talking about additive relations increased pupils’ sensitivity such that even when a relation had not been used in calculation, pupils were subsequently able to identify the additive relation and to talk about how it could be used.

Encouragement to notice in the two studies above took the form of teachers’ gesture aimed at drawing attention to particular recorded numbers and inscribed features, and, teacher questioning which was focused upon the mathematical features that were of particular relevance (Lobato et al., 2013; Voutsina and Ismail, 2011). The significance of carefully focused lines of questioning is confirmed by others. Strong affordances arise firstly through questions that frame the problem space or which identify the field that pupils should consider in responding (Gresalfi et al., 2012), and secondly through questions that offer, through the connection between questions, a provocation for pupil attention to focus on a particular feature or relationship (Towers and Martin, 2014). In this latter study findings reveal that teacher questioning interventions were productive in provoking awareness at specific stages of group activity. These were firstly in situations in which it appeared that there were properties or features of significance which individuals either had not noticed, or which had been noticed but not considered relevant and secondly where a group’s problem solving processes appeared to have stalled. In both cases the intervention consisted only of the posing of questions; the teacher did not take part in the subsequent discussion (Towers and Martin, 2014).

The use of resources and representations may also provoke productive noticing. Drawing on Bruner’s theory of representation (1966) Mason and Johnston-Wilder (2006), argue that the use of physical resources enables pupils to become aware of underlying mathematical structure. The value of enactive representation in supporting pupils’ construction of generality and reasoning is also documented (Marley and Carbonneau, 2014; Fyfe et al., 2014). With particular reference to lower attainers this theoretical assertion is supported by observational evidence that the
use of number equipment supports lower attainers to notice generality (Houssart, 2004), and to articulate relationships noticed (Foote and Lambert, 2011).

Using software that enabled pupils to work with a dynamic image, Hewitt (2009a) describes an attempt to deliberately focus pupil attention on viewing an expression as an object and working with it as such, rather than viewing it as a process to be carried out. Together with teacher questioning, software provided the opportunity for pupils to test out their predictions and supported the introduction of particular conventions of notation. In this study the triggering of surprise at the outcomes obtained served to stimulate pupils’ awareness of a new way of expressing the transformations undertaken using the software. Together these findings provide momentum for the use of such resources to support a range of awarenesses for all pupils.

Gattegno’s term, ‘forcing awareness’ (1987) suggests a greater intended control over what a learner becomes aware of than is maybe intended or possible. Powell (1998) considers that the term encompasses two meanings, what we intentionally do to ourselves in directing our attention, and what others may do for or to us in providing experiences and guiding our attention. In relation to the latter, authors agree that teachers cannot direct pupils’ awarenesses (Powell, 1998; Hewitt, 2001); awareness can only be accomplished by the individual pupil themselves (Powell, 1998). Rather, educating awareness draws on approaches such as directing or guiding pupils’ attention to particular mathematical features, helping them to reflect on the impact of their mathematical decisions and actions, and the use of particular lines of questioning (Hewitt, 2001). Teaching must, he argues, begin with the revealing of pupils’ awarenesses, requiring sensitivity on the part of the teacher in exploring these awarenesses and leading to the emergence of particular ideas or misconceptions that can be further explored using more open questions inviting prediction and the articulation of what is noticed (Hewitt, 2001).

The research evidence presented in this section suggests that a pedagogical focus on developing pupils’ awareness of particular mathematical ideas, relationships and properties, not only stimulates awareness in the moment, but enables these ideas to
become part of a ‘noticing repertoire’, influencing both how students attend and what their attention is focused on. This directly informs the intervention used in the current study in which the use of representations, encouragement and teacher questioning all contribute to the pedagogical focus on noticing. However, as I have noted, specific research data is limited. Furthermore, where it is available, with the exception of Hewitt (2009a), research either relates to pupils of secondary school age (Hohensee, 2016; Lobato et al., 2013; 2012), to studies not conducted in a classroom context (Voutsina and Ismail, 2011; Mulligan and Mitchelmore, 2009), or to studies in which awareness was not the analytic focus (Houssart, 2004). The current study thus contributes to understanding in this field through its classroom-based focus on promoting mathematical awareness in primary aged pupils.

2.2.5 Awareness and the lower attainer

The connection between a focus on awareness and the needs of lower attaining pupils has been alluded to above through reference to the work of several authors (Voutsina and Ismail, 2011; Foote and Lambert, 2011; Houssart, 2004). This connection is significant in the potential it offers for learning for this group of pupils and thus has clear relevance in focusing the current study. Here I draw together threads that relate to the needs of lower attainers and hypotheses drawn from research relating to how awareness may support this group.

Hohensee (2016) suggests that a focus on noticing had particular impact on pupils who were underachieving at the outset of his study. His results indicated that these pupils made greater gains in their awareness and use of the mathematical relations on which his study was based. This leads Hohensee (2016, p89) to hypothesise that noticing ‘may be a mechanism that could be leveraged to help students who are low achieving to catch up with students who are achieving at more proficient levels’. This suggestion is in line with that of both Mulligan (2011) and Voutsina and Ismail (2011), who also argue for the benefit of a focus on awareness for low attainers. The former suggests that without such attention to awareness, low attainers were more likely to focus on non-mathematical features of tasks and were less independently predisposed to look for or recognise underlying mathematical features such as
patterns and relationships (Mulligan, 2011). In explaining the success of their intervention Voutsina and Ismail (2011) suggest that their focus on awareness altered the goal of the task from solving to explaining, and that this was important in increasing pupils’ sensitivity to the additive relations that were the focus of the study. Where lower attainers are frequently exposed to tasks requiring recall and procedure (see 2.1), Voutsina and Ismail’s (2011) finding can be seen to be in line with Hewitt’s (2001) argument that a focus on awareness reduces the load on memory, bringing decision making to the fore.

In a further argument for addressing the needs of lower attainers through a focus on awareness, Hewitt (1997) advocates for a change in the way that mathematics itself is conceptualised. He argues that the traditional hierarchical view of mathematics in which a building blocks metaphor is used to conceptualise mathematics as increasingly complex and sophisticated sets of knowledge and skills that each build on an earlier stage is unhelpful. This, he argues, paves the way for setting and ability grouping practices since it purports to enable teachers to target learning at the appropriate ‘building block’ level of the pupil group. In an argument that builds on the ideas of Gattegno (1988; 1987), and which finds echoes in later work by Mason and Johnston-Wilder (2004), Hewitt (1997) argues that pupils have demonstrated understanding of fundamental mathematical ideas of sameness and difference, order, and inverse through their early childhood pre-school experiences. Their natural capacity to engage in this manner, to discern difference, to generalise and to look for pattern, pre-disposes all pupils to be able to engage with mathematical ideas (Mason and Johnston-Wilder, 2004; Hewitt, 2009b). Hewitt (2009b; 1997) argues that building a curriculum and individual lesson experiences on pupils’ ability to notice, to observe, to predict and to generalise, means that the content is accessible to all because engagement does not rely on previous stages learned.

Literature presented here makes a strong theoretical case for the potential benefit for lower attainers from a focus on noticing. In the following section I draw on the ideas discussed in the subsections above in considering how awareness will be identified and analysed in the current study.
2.2.6 Researching awareness: implications for the current study

Attempts to identify what pupils notice, attend to, or are aware of, are complicated by the partially internal nature of these constructs. If awareness is considered to be the event that occurs between not-knowing and knowing (Young and Messum, 2011) it is an event that is not directly accessible. Thus many awarenesses are beyond the reach of the observer, who can only work with what is publicly available (Barwell, 2003). Since language is a medium through which pupils both make explicit what is concerning them and seek to direct the attention of others, analysis of verbal interaction can serve to reveal patterns in what is publicly attended to as pupils work together (Barwell, 2005; 2003). Language can also show how pupils manage the shifts in attention between one focus and another. In addition to analysis of dialogue, Lobato et al (2013) and Hohensee (2016) analyse pupils’ gestures, diagrams and written inscriptions in order to enrich understanding of what pupils notice and attend to and use as the task progresses. Others, for example, Bolden et al (2015) have utilised eye tracking software to track the focus and length of gaze as pupils engage in mathematical tasks.

However, authors share the expression of caution about the way in which data can be used and the limits to the conclusions that can be drawn. Tracking of gaze permits inferences about the way in which pupils attend to task features and enables comparisons between this and the outcomes on these tasks, however Bolden et al (2015) stress that their data cannot be used to make assumptions about what pupils are thinking about. Barwell (2003) offers a similar caution in relation to the analysis of interaction, stressing that discourse analysis cannot reveal what pupils think, but only how they deal with their thinking through interaction. In the same vein, whilst awareness guides and informs actions and decisions, attempts to infer or intuit awarenesses from observed actions or from speech must be approached with caution (Mason, 2008a) and deliberate intent (Hewitt, 2001). Actions may, for example, suggest understanding of a mathematical idea when only competence in a procedure is secure; speech may incorporate the language of mathematical generality when the pupil is referring to an individual case only (Mason, 2008a). This notwithstanding, Hewitt (2001) argues that teachers can, with care, draw on their
own awareness in inferring the awarenesses underpinning pupils’ actions and speech.

Together these considerations deepen my understanding of the need for caution in inferring awareness from the speech and actions of focus pupils in the current study. Whilst I cannot know the multiple mathematical awarenesses that pupils bring to each mathematical activity, I can examine their actions and interaction to infer those that inform their initial approaches to the task. Further, through identifying changes and developments in their activity and interaction I can infer some new awarenesses that are developed as the task progresses. I do not claim to identify all the awarenesses that a pupil develops during an activity; I seek to infer those that provoke observable action and interaction. In working with pupil awareness in the context of this study, I therefore build on Mason’s (2008a) interpretation of awareness as a sensitivity in both detecting and responding to stimuli. My definition of mathematical awareness focuses on sensitivity to stimuli: ‘the detection of, or recognition of the applicability of, mathematical features, relations, or processes’. However, my analytic focus is on pupils’ response to the detection of these features as reflected in their action and speech. Awareness informs decisions and triggers actions (Mason, 2008a; Hewitt, 2001) but awareness is not the actions themselves (Hewitt, 2001). Building on this I begin analysis by noting the observable action and use this to infer the awarenesses that underpin the activity observed. Literature examined in the sub-sections enables me to identify specific types of awareness that might inform pupil action and interaction; each of these is outlined below.

Gattegno’s (1987) description of a four stage learning process and Mason and Johnston-Wilder’s (2006) identification of a manipulating – getting a sense of – articulating framework, together provide grounding for a focus on pupils’ approach to their tasks. In relation to mathematical activity and learning, these frameworks can be seen to incorporate a broad set of mathematical processes linked to enquiry and exploration. For example, Mason et al (2010) describe increasingly organised and deliberate trials through identification of random, systematic and artful specialising. Posamentier and Krulik (2009; 1998) identify a set of strategies including drawing images, constructing tables and fixing variables, amongst others, that support the
collation of information and thus the identification of pattern and underlying structures. These strategies are frequently used in mathematical enquiry to support a focused, systematic approach. Thus, my first awareness category is that of ‘awareness of mathematical processes’; inferred through observation of pupils’ actions, together with their dialogue as they engage in their paired mathematical activity. This category is included to support analysis of how mathematical processes influence the development of the task. It includes both identification of the initial approaches that pupils take as well as how changes and developments in their approach relate to the mathematical progress made.

The second awareness category is ‘awareness of mathematical pattern and property’. A mathematical pattern is defined as a ‘predictable regularity’ (Mulligan and Mitchelmore, 2009); mathematical property refers to the classes to which a mathematical object belongs (for example the numbers 2, 4 and 6 all belong to the class of even numbers). Mason (2003) identifies shifts in attention from whole to part and from part to part as an intrinsic aspect of stressing and ignoring. This analytic focus thus seeks to identify instances in which pupil attention might take the form of ‘discerning details’ and ‘perceiving properties’ (Mason, 2011) such that they become aware of properties of and patterns within the outcomes and recorded results that they have obtained. This focus incorporates both the identification of properties of individual numbers but also the identification of properties that are shared between a set of numbers. This latter aspect, the identification of commonality, of shared property or feature between different parts is significant for the move to generalisation (Towers and Martin, 2014) and for the understanding of structure (Mason, 2003). This renders it an important focus for analysis. However, Owens (2004) finds that lower attaining pupils experience greater difficulty in disembedding relevant parts from a whole and may require greater support in focusing attention productively; Mulligan (2011) similarly finds that low achievers do not appear to look for, and experience greater difficulty in identifying underlying mathematical similarity and difference. The particular challenge for lower attainers adds to the significance of this feature for the current study.
My third area of focus is on pupils’ ‘awareness of structure and generality’. Mulligan and Mitchelmore (2009, p34) define structure as ‘the way a mathematical pattern is organised’, noting that structure is expressed in the form of a generalisation, a relationship that is consistent within a defined domain. Mason (2003) connects properties and structure in identifying that structure emerges when a property can be separated from its particular example and considered independently. This analytic focus draws on the emphasis placed on awareness of structure for mathematics achievement identified by Mulligan and Mitchelmore (2009), together with the essential role of generality in mathematical activity (Mason et al., 2010; Mason et al., 2005; Mason, 2003). It also draws on Mason’s (2011) identification of ‘recognising relationships’ as a way of attending and seeks to identify instances in which pupils’ attention shifts to the consideration of relationships between task or mathematical features. This focus differs from the identification of commonality of property above in that it is concerned with the awareness of relationships between properties or features, and thus on the organisation underpinning the properties identified. In the context of this study, the term mathematical structure relates both to task specific structure, for example the relationships imposed by the constraints of the task, and to broader mathematical structure and generality. Evidence to support awareness of generality is drawn from speech and action, with attention to the caution expressed earlier in this section (Mason, 2008a; Hewitt, 2001).

My positioning of awareness as a necessary but not sufficient condition for reasoning raises a reasonable question as to the position of the boundary between awareness and reasoning. This question is maybe particularly pertinent in the case of awareness of generality since experience would suggest that generality is frequently articulated in the course of developing an argument. In the context of this study my positioning of awareness leads me to seek to infer the awareness that underpins the articulation of generality; the argumentative intent of the expression is also subject to scrutiny to evaluate the case that it contains.

Despite their different focus, the two categories of ‘awareness of mathematical patterns and property’ and ‘awareness of structure and generality’ bear some overlaps. Towers and Martin (2014), have noted that the identification of property is
connected with the development of generality; furthermore the structures embedded in the mathematics of the task will impact on numerical outcomes and thus the properties of these. In data analysis the distinction between these is drawn from what is being attended to at the time and how this supports inference as to the underpinning awareness. For example, if a feature of the outcomes is noted whilst examining a recording sheet, this is noted as an awareness of mathematical pattern and property. However, if, whilst manipulating a resource, a pupil notes an organisational feature, a possibility or a constraint, then I infer that the utterance is stimulated from this manipulation rather than from awareness of the results already obtained. In this case I would identify this as a structural awareness. This distinction is illustrated in later analysis chapters.

An additional category of awareness, that of ‘awareness of the rules of the activity’ is included in order that I can focus analysis on sections where it is evident from speech and/or action that pupils are engaging in the task as defined by the teacher and have not either misunderstood or decided to engage in an alternative task. Beyond signalling the starting point for analysis, the development of this awareness is not a focus for consideration. The analytic framework together with examples that enable inference of awareness in each category is set out in tabular form in 3.7.3.

2.3 Authority and influence in mathematics learning

Literature presented in 2.1 suggests that the use of mixed attainment groupings can provide valuable opportunities for mathematical learning for lower attainers. Such opportunities have been seen to arise from the exposure of the lower attainer to more complex mathematical ideas and challenges, and to the higher level thinking and dialogue of their higher attaining partners. However, a full consideration of the use of mixed attainment groupings must include closer examination of the detail of the interaction between pupils and of how this impacts on learning. This section thus considers literature that relates to the relations that develop between pupils in mixed groups. Specifically, I consider how an actual or perceived disparity between pupils in terms of their mathematical knowledge and understanding, or in their social status
may impact on nature of the interactions which take place, and how this may serve to either promote or inhibit learning for the pupils concerned.

2.3.1 Theoretical constructs in exploring authority relations

The term ‘authority’ frequently conjures ideas of power, control or dominance, with these ideas tending to focus on the bearer of authority and their ability to command or coerce (Gerson and Bateman, 2010). However, literature reflects wide agreement that for one person or party to be able to exercise authority, another person or party must accede or consent to, or otherwise recognise the legitimacy of this authority (Gerson and Bateman, 2010; Brubaker, 2012; Pace and Hemmings, 2007; Amit and Fried, 2005). Thus the term authority describes a ‘quasi-reciprocal’ (Weber and Parsons, 1964) relationship between individuals that exists by virtue of one person’s acceptance of the claim to legitimacy of the other (Gerson and Bateman, 2010; Brubaker, 2012). Importantly for Benne, (1970), to be considered an authority relation, rather than one in which power or control is exerted, the claim to legitimacy is associated with a particular field or sphere of activity only. Thus, for example, the authority of the teacher is granted in the field of education and the school only; similarly, a group of pupils might grant temporary authority to another pupil in order that a group task can be efficiently completed. Furthermore, such relations are a necessary part of the interdependence of human social existence, facilitating both learning and productive activity (Benne, 1970).

Seeking further detail of the grounds upon which either students or teachers successfully claim authority in the mathematics classroom, Gerson and Bateman (2010) drew on a grounded theory approach in analysing mathematical interaction between students and between teacher and students. Their resulting categorisation, comprising four main types of authority relations, provides me with a basis for further examination of how such relations impact on pupil learning. Their first main type, hierarchical authority, is a claim to legitimacy based on position. Within this category, ‘institutional’ authority is vested in the role of the teacher as a representative of the educational establishment; this authority may be temporarily ‘granted’ by the teacher to a pupil to lead a group, or to present an explanation to
the class. Here the authority granted by the teacher is to act in the role of the teacher in leading a discussion with the expectation that others in the group will comply with this (Gerson and Bateman, 2010). The remaining three categories are identified here and expanded upon in later subsections. Mathematical authority recognises that the discipline of mathematics acts as an authority; ‘expertise’ authority represents a claim to legitimacy through the possession of mathematical knowledge. The final category of performative authority reflects a claim based on the ability to engage and enthuse.

The work of Gerson and Bateman (2010) and Benne (1970) provoke important considerations in the context of the current study. Firstly, it is evident that authority relations can be based on a variety of claims to legitimacy. Secondly, in the context of mathematics education, such relations exist between students as well as between teacher and pupil (Gerson and Bateman, 2010). Thirdly, where authority relations are mutually established in order to enable efficient and successful working they can be productive (Benne, 1970). Lastly, since authority relations are as much to do with those who accept the authority claims of others as they are to do with those who come to bear authority, where such relations are unproductive, routes to greater equity require situations that empower the traditional acceptors of claims to legitimacy (Gerson and Bateman, 2010).

The following subsections draw on and further examine three of Gerson and Bateman’s (2010) categories. Firstly, I further examine the notion of ‘mathematical’ authority, the authority of the discipline itself; secondly, I explore the way in which the positioning of pupils with ‘expertise’ authority impacts on learning for the group. Lastly, I examine how social status impacts on interaction, a focus stimulated by, but not precisely equivalent to Gerson and Bateman’s ‘performative’ authority.

2.3.2 Mathematics ‘discipline’ authority relations and pupils’ mathematical learning

The notion that pupils may engage in authority relations with mathematics arises from the assertion that the discipline of mathematics acts as an authority (Gerson and Bateman, 2010; Boaler, 2002; 2003). Drawing on the accepted rules, principles
and the ways of working of the discipline reflects an acceptance of the authority of these practices in achieving mathematical aims.

It is important for pupils to have opportunity to exercise disciplinary authority and make claims based on their knowledge of mathematical practices (Boaler and Sengupta-Irving, 2016; Boaler, 2016). However, it is also important that this is balanced by the opportunity for pupils to draw on their own ideas, claim ownership of solutions (Povey, 1997), contribute insights or conjectures and make decisions about ways of working and recording. Within recent literature this is frequently referred to as the exercising of individual agency (Grootenboer and Jorgensen, 2009; Boaler, 2002; 2003). Research in this field is clear in its conclusion that such opportunity is a significant part of developing an identity as a mathematical thinker. Conversely, restriction of these opportunities, in favour of relying solely on the discipline of mathematics, or solely on the teacher’s evaluation of whether the requirements of the discipline have been satisfied, positions pupils as receivers of knowledge rather than as active agents. This is suggested to limit mathematical engagement and thinking and to be associated with negative subject views and a difficulty in applying ideas in novel situations (Boaler, 2002).

Gresalfi and Cobb (2006) agree, concluding that the authority relations in a classroom are inextricably linked to the way in which pupils interact with mathematical ideas and with each other. The arising recommendation that authority relations in the classroom should thus reflect greater distribution of authority such that students more frequently make claims to knowledge based on their own mathematical justification, rather than relying on the teacher’s evaluation of their responses has wide support (Amit and Fried, 2005; Gerson and Bateman, 2010; Gresalfi and Cobb, 2006; Grootenboer and Jorgensen, 2009; Boaler, 2016; 2003; 2002). Drawing from Pickering (1995), Boaler (2003; 2002) proposes that achieving a balance of the use of mathematics disciplinary practices, rules and procedures, together with the use of one’s own decision making insight and conjecture can be seen as engaging in a ‘dance of agency’. That this is a desirable and productive situation to achieve is echoed by Grootenboer and Jorgensen (2009).
The rationale for the recommendation of greater distribution of authority extends beyond an aim of merely increasing participation. Gresalfi and Cobb (2006) contend that in classrooms where authority is distributed such that pupils are both invited and expected to be involved in decision making about methods and processes, to make mathematical suggestions and evaluate the mathematical validity of the contributions and ideas of others, they will engage at a deeper level mathematically. Causality is further argued through the suggestion that productive engagement in mathematical discourse develops because of this increase in student autonomy (Kosko and Wilkins, 2015). In short, “having to ‘convince’ someone else requires higher standards of argumentation than simply being accountable to a teacher who gives confirmation of accuracy” (Gresalfi and Cobb, 2006, p52). Further support for this assertion is derived from a converse example: Amit and Fried (2005) observed that pupils’ frequent unquestioning acceptance of their teacher’s mathematical authority limited the development of the pupils’ own mathematical thinking, reflection and self-reliance.

Whilst literature examined here is unequivocal in the recommendation of the distribution of mathematical authority, two considerations emerge. Firstly, the task set must provide opportunities for decision making relating to approach, the making of conjectures, and evaluating the quality of justification, in order that disciplinary authority can be meaningfully shared. Secondly, all pupils in a mixed group need to benefit from the opportunities presented by this distribution of authority. Indeed, a relationship between distributed authority relations, task type, and the opportunity for all pupils to exercise agency in contributing ideas is explicitly identified by Mueller et al (2012). The identification of this relationship thus further strengthens the rationale for the use of open structured tasks discussed in the context of mixed attainment groups in 2.1.4. The use of such tasks, alongside considerations of the need for all pupils to benefit from the opportunities that they afford is further discussed in the following section.
2.3.3 Mathematics ‘expertise’ authority relations and pupils’ mathematical learning

Langer-Osuna (2016) asserts that the study of authority relations between pupils is fundamental to understanding how mathematical solutions are constructed when pupils work together. The influence of such relationships is particularly noted by Langer-Osuna (2016) who extends the remit of the ‘expertise’ authority identified by Gerson and Bateman (2010). She argues that in addition to mathematical authority between pupils manifesting as ‘intellectual’ authority, interpreted as being perceived as a credible source of information relative to the task in hand, this authority becomes connected to a ‘directive’ authority, the power to direct the activities of others within the task (Langer-Osuna, 2016). This is not an isolated finding. Esmonde (2009b) also found that the positioning of an expert frequently lead to other pupils within the group deferring to this expert in matters relating to task organisation and mathematical accuracy.

Research also points to the limiting nature of dialogue where such relations exist, restricted largely to the transmission of information, rather than being focused on debate, reasoning or explanation (Amit and Fried, 2005; Esmonde, 2009b), and with little challenge presented to the solutions proposed by the ‘expert’ (DeJarnette and Gonzalez, 2015). Pupils positioned as mathematically credible are more likely to have their ideas listened to and accepted (Langer-Osuna, 2016). Conversely those positioned as ‘novices’ are frequently ignored in conversation and decision making, tending to be given, and to comply with, instructions from others, and either watching or engaging in largely procedural aspects of the task (Wood and Kalinec, 2012; Esmonde, 2009b). This expectation of greater competence is mutually constructed; those with higher perceived academic ability expect, and are expected by others to be more competent at the task, regardless of whether their perceived academic ability is specifically linked to the task in hand (Cohen and Lotan, 1997). Domination of the verbal interaction and lack of collaboration arising from such expertise authority relations means that those with recognised or perceived mathematics expertise have greater opportunity to engage in mathematical thinking and activity and thus gain more from the task (Esmonde et al., 2009; Esmonde, 2009b; Cohen and Lotan, 1997; 1995; Esmonde and Langer-Osuna, 2013). Such
differences in opportunity for interaction serve to generate further differences in learning outcomes (Cohen and Lotan, 1997).

Whilst ‘experts’ are not always overtly or explicitly identified, there is wide agreement that in a mixed group, pupils know who to ask for help or who has more mathematical credibility (Amit and Fried, 2005; Esmonde, 2009b; Webb and Mastergeorge, 2003; Engle et al., 2014; Langer-Osuna, 2016). At all levels of mathematics learning such expert knowledge can be persuasive (Inglis and Mejia-Ramos, 2009). However, whether help sought is beneficial for the help seeker’s learning is contingent on particular factors. Where a pupil can be specific about the help they require, and persistent in seeking it, then assistance from an ‘expert’ peer can not only improve achievement at the time but result in longer term learning (Webb and Mastergeorge, 2003). Their findings point towards responses from peers that identified key conceptual, rather than merely procedural, explanations. Conversely when help sought was unspecific, such as general appeals of being stuck or confused, without detail as to the nature of this difficulty, peers were less able and willing to assist. Further, when purely solutions were sought and given, little learning, either in the short or longer term, was evident (Webb and Mastergeorge, 2003).

Esmonde et al (2009) found that those positioned as experts felt a tension between wanting to include others and wanting to get work done, that they did not always welcome the positioning of themselves as expert and felt that being part of a mixed group slowed their progress. Lower attaining pupils in the same mixed groups recognised that their questions slowed others down and felt a pressure to say that they understood when they did not. Pupils in this study also pointed to social factors that affected the equitable functioning of groups, an issue addressed later in this chapter, raising questions for the authors in relation to the value of mixed groups where social and mathematical authority relations appeared to limit opportunities to learn (Esmonde et al., 2009). Whilst some of these findings are consistent with those of Hallam et al, (2004b) the latter also found strong support from primary aged pupils for the use of mixed groups due to the opportunity to learn from and motivate each other, to get to know a wider range of pupils and to avoid stigmatising low attaining pupils.
The balance of evidence presented thus far in this section indicates that expert/novice authority relations, except in specific circumstances (Hallam et al., 2004b; Webb and Mastergeorge, 2003) do not serve the novice well. Such relations do not appear to support collaboration or discussion or enable the lower attainer to construct ideas for themselves (Amit and Fried, 2005; Esmonde, 2009b). This appears to run counter to the evidence presented in 2.1.4, which pointed towards learning gains for lower attainers in mixed attainment pairings. Whilst many factors will be involved in the differences in outcomes, the structure and affordances of tasks emerged as a significant factor in promoting equitable participation (Boaler, 2016; Boaler and Sengupta-Irving, 2016; Boaler and Staples, 2008; Engle and Conant, 2002; Alexander et al., 2009; Turner et al., 2011) and is worthy of further consideration in the context of the findings above.

Prevalent in the discussion in the current section has been the notion of unidirectional help seeking; research reports (Amit and Fried, 2005; Webb and Mastergeorge, 2003; Esmonde, 2009b) provide no evidence of the extent to which tasks afforded opportunities for lower attainers to make a valuable contribution beyond performing procedures. This is significant given that a key factor in promoting equitable engagement is the opportunity for reciprocal help seeking more frequently afforded by an open structured task (Boaler, 2016; Boaler and Staples, 2008; Cohen and Latan, 1995). As noted in 2.1.4, these tasks are characterised by their investigative or problem solving context and opportunities for multiple ways of working, an accessible starting point and intellectual challenge (Boaler and Staples, 2008; Boaler, 2016). Indeed, where authors focusing on authority relations have drawn on the use of tasks of this type, findings point towards mixed groups operating successfully. For example, in separate studies, increased collaboration and greater discussion and reasoning were evident when despite attainment differences between pupils, no pupil was positioned as an expert (DeJarnette and Gonzalez, 2015; Esmonde, 2009b). Here, the more dynamic sharing of authority was characterised by the creation of opportunities for all to talk and to engage in challenge (DeJarnette and Gonzalez, 2015).
Further support for the importance of task structure comes from the finding that single answer tasks and tasks that place high value on speed and accuracy in completing mathematical procedures privilege higher attaining pupils (Boaler and Staples, 2008; Chizhik, 2001). These tasks narrow the scope for dialogue to assistance with procedures and the checking of accuracy (Boaler and Staples, 2008) and more frequently lead to task dominance by positioned ‘experts’ (Turner et al., 2011). In these situations lower attaining pupils recognise that fast pace and a competitive environment limit their opportunity to participate (Black and Varley, 2008). Thus, whilst it has been argued (Amit and Fried, 2005; Esmonde, 2009b) that the presence of mathematical authority relations frequently limits dialogue, the role of task affordances in establishing or promoting these authority relationships cannot be discounted and may account for some of the findings above.

However, it would be naive to assume either that open structured tasks preclude the possibility of authority relations emerging or that authority relations are inevitable in the absence of such tasks. Indeed research drawing on the use of open tasks has noted that whilst such mixed groups present opportunities for pupils to share ideas and act as a resource for each other, they can also suffer from issues of dominance, with lower achievers being marginalised and being afforded limited access to either the instructional materials and resources for the task, or to the interaction that takes place (Cohen et al., 1999; Cohen and Lotan, 1997). In order to address such inequity these authors promote the use of variations of ‘multiple ability treatments’ in which perceived academic status differences between pupils are deliberately tackled by the teacher through the public praising of specific task related competencies demonstrated by lower status pupils. Such intervention serves to raise the status of pupils through identifying the range of intellectual contributions that can be made to the successful completion of a collaborative task (Boaler and Staples, 2008; Cohen et al., 1999; Cohen and Lotan, 1997). Thus, whilst the use of open structured tasks provides greater affordances for equitable participation and learning and reduces the potentially negative impact of the emergence of ‘expertise’ authority relations, careful teacher intervention is required in order to ensure that lower status pupils are enabled to realise these affordances.
2.3.4 Group interaction and pupils’ mathematical learning

That authority has an element related to personality and performance has been recognised by those categorising and describing this relation. Gerson and Bateman’s (2010) category of ‘performativ authority’ together with Weber and Parsons’ (1964) earlier identification of ‘charismatic authority’, both speak to the use of persuasion and in the former case gesture, humour, body language and movement to influence others, to support an argument or indeed deflect attention. However, Langer-Osuna notes that beyond such performative features, pupils’ task based interaction frequently “falls prey to issues of status” (2016, p108). Research shows that pupils use the interaction within their mathematics activities to achieve social ends, to enact power struggles and build social identities (Esmonde et al., 2009; Langer-Osuna, 2016; Wood and Kalinec, 2012). Such research importantly reveals how this interaction can limit or promote opportunities for particular pupils to engage mathematically.

Several authors have noted that the influence pupils exert over the direction of a task is not always related to their mathematical attainment or to the quality of their ideas or suggestions. Barnes (M, 2005) found that higher social status students were more powerful than their lower status peers in that their ideas were more likely to be acted upon, regardless of their mathematical validity, and that they were frequently instrumental in closing down a discussion or avenue of exploration, again regardless of the mathematical validity of this decision making. Engle et al (2014) similarly reports instances of pupils’ undue influence promoting misleading or poorly formed ideas and Langer-Osuna (2016) notes how a pupil’s social status can lead to directive authority over the processes of a task, in itself leading to the positioning of the pupil as having an intellectual authority that is not warranted.

Further evidence for the ineffectiveness of mathematical competence alone in influencing task direction is found in studies which draw from pupils’ own accounts of the dynamics within group work. Both Reay (2006) and Esmonde (2009) find that pupils identified by others as of ‘high ability’ needed also to be popular in order to be taken account of in group work. One pupil noted that for groups to work well
there needs to be someone who is good at maths and at the same time is ‘broadly liked’ (Esmonde et al., 2009, p32). Allied to this, Barnes (M, 2005) found that the mathematically sound suggestions of lower social status group members required the assent of more popular peers in order for them to be acted upon. In what might be described as an extreme case, Heyd-Metzuyanim and Sfard (2012) report a situation in which group members repeatedly rejected the cogent argument of a higher attaining pupil in favour of working hard to make sense of the much less coherent argument of more popular other. In both this and a separate study (Langer-Osuna, 2011), the negative positioning of the higher attaining pupil arose in response to the teacher’s positioning of the higher attainer as ‘smart’ and worthy of being listened to. The challenge to this granted authority (Gerson and Bateman, 2010) in the form of increasing marginalisation and reduced mathematical engagement of the targeted pupil (Langer-Osuna, 2011), provides a salutary lesson for teachers considering such practices.

At first sight, evidence thus far would suggest that social status is frequently a more powerful influence than is rational argument in pupils’ interaction during mathematical tasks, and that intellectual capability does not appear to raise a pupil’s social status or to necessarily grant influence in the context of a mathematical task. This appears to run counter to research presented in 2.3.3, which identified that in many situations, high attaining pupils were frequently positioned as experts and dominated task interaction. In order to reconcile these two apparently opposing sets of findings, it is worth recalling that authority relations only meaningfully exist when a claim to legitimacy is accepted by others (Gerson and Bateman, 2010). The examples cited above thus reveal that in interaction between pupils, a claim based on mathematical expertise may not be accepted if the claimant is at that moment, unpopular for one or more reasons. Indeed Engle et al (2014) stress that the mathematical quality of an argument can, and may often prevail, but seek to show how and why this may not always be the case. Both Engle et al (2014) and Heyd-Metzuyanim and Sfard (2012), note that their examples are drawn from relatively extreme and explosive instances of interaction between pupils. Whilst extreme they
nevertheless are included here as revelatory of the different influences at play in the progress of a task.

The way in which a pupil is positioned through interaction can significantly impact on their opportunities for mathematical engagement and opportunity to learn (Wood and Kalinec, 2012; Wood, 2013; Heyd-Metzuyanim and Sfard, 2012; Langer-Osuna, 2011). For example, Wood (2013) identifies different types and levels of engagement within the same lesson when a pupil was positioned by his peers and by the teacher in three different ways. When positioned by his teacher either as a student or as an explainer, the pupil, a relatively low attainer, was able to engage with mathematical ideas, and showed evidence of conceptual development through his discourse. Conversely, interaction with a higher attaining peer positioned this pupil as a ‘menial worker’, during which time engagement consisted only of following low level instructions made by the peer. No evidence of mathematical engagement was present during this time.

Whilst this example may appear to do no more than illustrate the findings presented earlier in relation to dominance of task interaction and engagement by higher attainers (Esmonde et al., 2009; Esmonde, 2009b; Cohen and Lotan, 1997; 1995; Esmonde and Langer-Osuna, 2013), the detail of the analysis permits important insights into the way in which this occurs. What is revealed is that interaction that is not task related, labelled ‘identifying’ talk by Wood and Kalinec (2012), and including talk about a person’s actions, their behaviour or about their qualities as an individual, can serve to either include or exclude individuals from particular episodes or aspects of the interaction. It is this that can impact on the way in which pupils engage mathematically, both at the time and subsequently.

Sfard (2002) acknowledges the existence of two intentions contributing to mathematics classroom communication, the object level and the meta level. The former is directly related to the particular activity, in this case the mathematical task; the latter is concerned with the relationship between participants, the way that interaction is progressing. Wood and Kalinec, (2012) assert that both separately contribute to, and are constituent of, learning, as both are part of the group.
interaction that produces or constrains mathematical learning outcomes. Sfard (2002) notes a frequent tension between the two types of intentions; in mathematical activity participants are simultaneously attempting to respond to mathematical content and negotiate their stake in the interaction. Sfard’s (2002) analysis noted the lower attaining pupil prioritising attempts to respond and attend to his partner’s utterances. Conversely, the higher attaining pupil’s focus was on vocalising his own mathematical thinking rather than communicating with his partner.

More recent research examining the way in which pupils discursively negotiate and re-negotiate their own and each other’s influence on the direction of mathematics and science tasks has led to the development of an ‘influence framework’ (Langer-Osuna, 2016; Engle et al., 2014) which sees influence as arising from four connected factors. These factors are, firstly, the perceived intellectual merit of specific contributions, secondly, the degree of intellectual authority that the pupil currently holds in the group. The third factor, the pupil’s access to the conversational floor, relates to the opportunities a pupil has to influence the direction of the task through speech and non-verbal interaction; lastly their degree of spatial privilege refers to the way in which each pupil is visually attended to when speaking. This latter can be affected by seating position and is seen as having an indirect effect on the way that influence is constructed (Engle et al., 2014). Together these factors represent each individual’s influence as dynamic, shaped over time and over multiple interactions.

Langer-Osuna (2016) also notes the importance that resources and artefacts can play in the influence that pupils can generate. Indeed, building on the ‘influence framework’, Dookie and Esmonde (2012) identify the particular role of resources in mediating the way in which influence in a mathematics task is constructed. Here each aspect of the influence framework was seen to be affected by pupils’ access to and control over the artefacts that were central to the task. For example, access to the key resources enabled pupils to carry through their ideas without discussion, effectively bypassing the need for any evaluation of the intellectual merit of the ideas and establishing an intellectual authority by virtue of being unchallenged. Similarly, control over the resources enabled easier access to the conversational floor and,
since pupils tended to centre their attention towards the resource, also bestowed a greater degree of spatial privilege (Dookie and Esmonde, 2012). These authors stress that whilst analysis of interaction reveals some aspects of the way that influence is exerted, close analysis of action is required to more fully understand the factors contributing to the involvement and marginalisation of pupils in group work (Dookie and Esmonde, 2012).

The studies reviewed in this section present mathematical learning as emerging from an interplay of social and mathematical interaction. Significantly it is apparent that mathematics expertise or intellectual authority forms only one factor in the development of influence in the social context of the classroom. The following section presents the implications arising from this review in shaping analysis in the current study.

2.3.5 Summary and implications for the current study

Research presented in 2.3.2 has identified the importance of a distributed disciplinary authority in providing opportunities for pupils to exercise individual agency (Grootenboer and Jorgensen, 2009; Boaler, 2002; 2003) during mathematical activity. Given that the current research introduces a pedagogic intervention which sets specific expectations for the nature of pupils’ engagement, analysis of these expectations will be significant in understanding the impact of the pedagogic focus on pupils’ subsequent activity and interaction. In this analytic focus, I examine the expectations, obligations and entitlements (Gresalfi and Cobb, 2006) placed on the pupils as expressed in the teachers’ introductions to the task and in their interventions at pupils’ tables and in plenary sessions at different times during the task.

Literature examined in 2.3.3 and 2.3.4 has noted the frequently detrimental impact arising in situations where expertise authority relations are manifested but has also demonstrated that influence is constructed through multiple interactions in which factors other than the intellectual merit of an idea or the intellectual standing of an individual are at play. This study seeks to examine the way in which action and interaction construct opportunities for lower attainers to contribute to activity and
learning in their pair, and I draw on aspects of the framework for analysing the
dynamic construction of influence (Langer-Osuna, 2016; Engle et al., 2014) in
identifying three analytic categories. The first is the perceived intellectual merit of an
idea. Engle et al (2014) suggest that ideas perceived as having merit can be directly
influential in a task, but that the greater the level of perceived authority a pupil holds,
the lower the requirement to provide well-founded or reasoned ideas. The reverse
is true for those with lower perceived authority. Here I evaluate the perceived
intellectual merit of suggestions and mathematical ideas proposed by the lower
attainer through the nature of the responses received from their higher attaining
partner. This focus is closely aligned to Gerson and Batemans’ (2010) category of
mathematics ‘expertise’ authority in that it addresses both the making of a claim to
expertise and the response to it, constituting either an acceptance or a rejection of
the claim made.

The analytic focus on verbal interaction to evaluate perceived intellectual merit of
ideas, is complemented, in line with the recommendation of Dookie and Esmonde
(2012), by analysis of pupils’ actions as they engage with the task. Through this I aim
for deeper understanding of how a mathematical idea expressed as a claim to
expertise may or may not become influential. Since the mathematical activities in
which pupils are engaged all involve the use of resources, I focus on the role of
artefacts in mediating the influence that pupils exert (Dookie and Esmonde, 2012;
Langer-Osuna, 2016). Here I draw again on the influence framework (Engle et al.,
2014) in my second category of spatial privilege. Here I use evidence of body position
and eye contact in examining how pupils attend to each other’s contributions and to
their activity with the resources.

My third focus is on the lower attainers’ visual and physical access to these same
resources. This is informed by Engle et al’s (2014) notion of access to the
conversational floor, but with a focus on pupils’ capacity to influence the task
through non-verbal rather than verbal means. This again reflects the centrality of
resources in each of the activities of the study. I have termed this third analytic
category ‘access to the activity space’.

64
The analytic framework setting out examples of action and interaction which align to each analytic focus is set out in tabular form in 3.7.3.

2.4 Aims and Research questions

This chapter has examined literature relating to the three main areas of provision for lower attaining primary school pupils, mathematical awareness, and mathematical authority relations. Together these are the elements that comprise my conceptual framework; it is the relationship between these components that the study seeks to examine.

I have argued that I view mathematical awareness as a necessary but not sufficient condition for mathematical reasoning. The development of pupils’ mathematical awareness, to support the construction of mathematical reasoning, is the focus of the classroom intervention that forms this study. The specific focus on lower attaining pupils has been justified through examination of their needs and of their capacity to engage in mathematical reasoning. Evidence to support the use of mixed attainment pairings presented in this chapter provides a rationale for the use of this pairing structure in this study. Here it is important to note that the use of this mixed attainment pairing constitutes a substantive internal relation (Danermark, 2002); i.e. in a mixed attainment pair each pupil’s attainment status is relative to that of their partner. The significance of this is that such relations have the potential, in critical realist terms, (see 3.1) to be generative of behaviours arising from this relationship. In the case of mixed attainment pairings, the generative powers arising from the mixed attainment relationship may result in an authority relation that works to disadvantage the lower attainer. Literature examined earlier in this chapter has identified such disadvantage arising for low attaining pupils in some scenarios.

My initial hypothesis is that without intervention, the structure of the mixed attainment pairing may stimulate the operation of an authority relation that disadvantages the lower attainer. This is informed both by the literature examined and by observations that I had undertaken as part of earlier stages of my doctoral study. As part of the current study, preliminary observations were also conducted to
further explore this hypothesis in the context of the three focus classrooms in which the study took place. The outcomes of these preliminary observations were used to shape the research questions presented below; analysis of the data arising from these observations is discussed in chapters 4 and 5.

2.4.1 Research aims

In conjunction with the outcomes of the preliminary observations, two hypotheses presented by researchers significantly influence the current study. These are, firstly, that a pedagogical focus on awareness presents particular learning benefits for lower attainers (Hohensee, 2016) and, secondly, that such a focus may operate to level the playing field, so to speak, in rendering mathematical content accessible to all (Hewitt, 2009b; 1997).

Building on this I further hypothesise that such a focus may militate against the emergence of authority relations that disadvantage the lower attainer. It may thus be an avenue through which lower attainers, as the traditional acceptors of others’ claims to expertise authority (Gerson and Bateman, 2010), can be empowered. This research explores these hypotheses through a classroom-based study in which a pedagogical focus on awareness seeks to enable lower attainers to make valuable mathematical contributions to the task, and to enable pair interaction to construct equitable opportunities for both pupils to engage and learn. The two aims guiding the research are as follows:

1. To contribute practical and theoretical understanding of the role of awareness in primary lower attainers’ mathematics learning and make recommendations for primary practice

2. To contribute to understanding of the relationship between pupil interaction in a mixed attainment pair and opportunities for the lower attainer to contribute mathematically
2.4.2 Research questions

As with the research aims, the research questions are informed by literature reviewed in this chapter, and by the outcomes of the preliminary observation. The research questions entail a close focus on the detail of pupil action and interaction as they engage in mathematical activity together. This detailed focus on a small number of pupils enables the study to contribute to the call for mathematics education to achieve a ‘greater understanding of its nuances’ (Boaler, 2003, p3) in order to understand the relationship between particular teacher decisions and pupil learning.

Four research questions are identified which seek to address the aims above. The first supports exploration of the nature of lower attaining pupils’ mathematical awarenesses:

1. In what ways does a pedagogical focus on noticing impact on the nature of lower attaining pupils’ mathematical awarenesses?

I have noted that the contribution that lower attainers can make to a paired task is not well understood or documented; the specific focus on lower attainers’ contributions in the second research question seeks to shed light on the role that lower attainers can make to such mixed pairings:

2. How do lower attainers’ mathematical awarenesses contribute to task progress for the pupil pair?

Research questions 1 and 2 address aim 1.

Literature reviewed has also raised cautions relating to the potentially negative interactions that working with a higher attaining peer may have for the participation and learning of the lower attainer. This gives rise to the third research question. This question focuses attention on the activity and interaction between pupils and enables exploration of authority operating within the pupil pairs:

3. In what ways does a pedagogical focus on noticing influence the nature of the interaction between lower attaining pupils and their higher attaining partners?
Lastly, literature identified the significance of the distribution of disciplinary authority in contributing to the way in which pupils interact with each other and with their mathematical tasks and this gives rise to the final research question:

4. In what ways does a pedagogical focus on noticing influence the nature of the expectations for engagement and activity conveyed through class teachers’ introductions and questions?

Research questions 3 and 4 address aim 2.

These research questions inform the design of the research study, its data collection and analytic methods. The aim of empowerment of lower attaining pupils through the focus on awareness reflects the ontological position of critical realism adopted in this research. This ontological position together with the way in which the methodology of the study permits the research questions to be addressed is discussed in the following chapter.
Chapter 3: Methodology

This study aimed to explore the impact of a pedagogical focus on noticing on the development and use of lower attainers’ mathematical awareness and on the nature of pair interaction.

The study took place in three KS2 classrooms (two year 6 and one year 4) in three different schools over the course of one academic year. Each of the focus lower attainers was working at below age-related expectations. For all lessons that were part of the study they were paired with a higher attaining partner who was working at above age-related expectations. Brief profiles of the schools, teachers and pupils reported in this thesis are included in appendix G; processes involved in the recruitment and selection of teachers, and of pupils, are outlined in 3.8.1 and 3.8.2 respectively.

This chapter details how this aim was operationalised in the planning stage, in the classroom and through the approach to analysis. The chapter begins with discussion of the ontological perspective that has informed the research at every stage and then focuses on how this ontological perspective provides a lens through which the social relations operating in the classroom might be viewed. Following this, I outline the chosen approach of Education Design Research and articulate how this is consistent with the ontological perspective adopted in the study. The subsequent three sections (3.4, 3.5 and 3.6) address the specifics of the study design, preparation for the study, and the intervention activities respectively. I then detail the approach to analysis in section 3.7; the final section of the chapter (3.8) addresses ethical considerations.

3.1 Ontological perspective

Critical realism provides the philosophical perspective from which this research is undertaken; the stratified nature of social reality articulated by Bhaskar (2008; 1998; 1989) enabling powerful insights in understanding and explaining the social world. This perspective provides the lens through which I view and interpret the operation of authority relations in the mathematics classroom and underpins the focus on lower attaining pupils as the following paragraphs explain.
Maton (2001) suggests that critical realism offers educational research an alternative to what he describes as the ‘entrenched positions’ based on empiricism and idealism frequently evident in educational discourse. This entrenchment is reflected in policy decisions with far reaching impact built on unfounded empirical assertions and closed system assumptions (see 3.2) on the one hand, with research based on idealism and driven by emphasis on individual experience and judgemental relativism on the other (Shipway, 2011; Scott, 2010; Maton, 2001). Shipway contends that critical realism offers a more balanced perspective, uniquely recognising the “interaction of the real environment with the causally efficacious interior world of the individual agent” (2011, p176). Critical realism’s explicit focus on ontological depth provides a counter to the claim that education research frequently lacks overt philosophical underpinning (Burnett, 2007; Scott, 2010) and leads Maton (2001) to assert that educational research is in particular need, and is ready for, the examination that a critical realist perspective offers.

In describing the operation of the social world, Bhaskar asserts a reality consisting of three overlapping domains of the empirical, the actual and the real (Bhaskar, 2008; 1989; 1998; Bhaskar and Lawson, 1998). Empirical reality consists of that which we directly experience; beyond this is the domain of the actual which comprises the experiences of the empirical world together with the events which give rise to them. Here Bhaskar stresses that not all events are directly linked to an experience; events may occur that we do not or cannot perceive and which our current understanding of the world does not permit us to interpret. To suggest otherwise, i.e. that there no ‘actualities unperceived’ (Bhaskar, 2008, p32), would be to suggest that our current knowledge represents the entirety of the state of the world; this is an assertion we are not in a position to make. The third domain, the real, incorporates the previous domains of events and experiences, but is particularly concerned with structures and mechanisms, and with the causal powers which arise from these (Bhaskar, 2008; Collier, 1994). For Bhaskar, the world ‘consists of mechanisms not events’ (2008, p47); the combination of the generative powers of these mechanisms is what creates the events and experiences of the world.
Structures operate independently of our knowledge and understanding of them. However, in the social world these structures are not independent of our existence (Hartwig, 2007), and thus a causal interdependence exists between social science and its subject matter that is not mirrored in enquiry in the physical world (Bhaskar, 1998; 1989). These structures, for example, the family, education, or democracy, both depend upon and pre-suppose social relations, (for example that between teacher and pupil, parent and child). Bhaskar (1989), argues that these social relations are themselves structures as they pre-exist any one individual’s encountering of them.

In the context of the current study, the authority relations discussed in chapter 2 that may exist between pupils are viewed as social relations, or structures, influencing the events and experiences of the classroom. As has been argued previously, authority relations are reciprocal; a claim to authority by an individual in a particular sphere cannot meaningfully exist without another who accepts this authority. In critical realist terms this represents a substantive, internal connection (Danermark, 2002) between individuals. This internal connection is a generative power, with the potential to stimulate actions and behaviours from each related individual by virtue of, and arising from, the relation that exists between them. Authority relations are thus an example of a causal relation.

More than establishing a distinction between the three domains, Bhaskar asserts a stratification of reality which enables more ‘adequate accounts of the world’ (Hartwig, 2007, p116) in that events can be explained vertically by reference to the mechanism and structures upon which they depend, by which they are influenced and without which they would not exist (Archer, 1998; Hartwig, 2007). Without this stratified view of reality, the three domains are effectively ‘collapsed into one’ (Bhaskar, 1998, p15) resulting in horizontal explanations of one event by reference to another.

Bhaskar argues that acceptance of social structures as real is a fundamental and necessary condition for meaningful study within social science. Indeed, it is these structures, rather than the phenomena they produce, which are the object of enquiry (Archer, 1998; Bhaskar, 1998; 1989; 2008). However, whilst they may be the object
of study, Bhaskar’s critical realism is unequivocal in relation to what constitutes meaningful study: understanding society is not an end in itself, the purpose of social science is to effect emancipation (Shipway, 2011; Bhaskar, 1989). In the context of the current study, this is embodied by the aim of empowerment of the lower attainer to contribute meaningfully to mathematical activity and to access challenging mathematical ideas alongside their higher attaining partners.

3.2 Critical Realism and the social world of the classroom

Critical realism’s perspective on the interplay between the individual agent and societal structures is a defining feature of this ontological perspective and impacts directly on research in the social world, for example that of a primary mathematics classroom.

Critical realism subscribes neither to the Weberian view that society can be reduced to the actions of individuals, nor to structuralism which places individuals at the mercy of society (Bhaskar, 1998; Scott, 2010; Danermark, 2002). Instead both structure and agency are viewed as entities each having distinct properties and powers; neither can be reduced to, or wholly explained in terms of, the other.

A distinct property of social structures is their anteriority; whilst they arise as a result of human activity, each of us is born into a world where these structures already exist (Carter and New, 2004) and within which constraints and enablements for particular types and modes of behaviour and interaction are embedded. All human agency is enacted within the context of, and is shaped by, social structures (Carter and New, 2004; Shipway, 2011; Archer, 1998).

However, human agency also has distinct, ‘sui generis’, properties and powers: our self-consciousness, capacity for reflexivity, cognition and emotionality, means that whilst we are influenced by social relations, our actions are not wholly determined by them (Carter and New, 2004; Archer, 1998). Humans are ‘causal agents in a world of other causal agents’ (Bhaskar, 2008, p215), and we act intentionally.
The complex interplay between the operation of pre-existing structure and human agency gives rise to the explicable, but unpredictable nature of the social world. Whilst the operation of social structures shapes the range of actions likely to occur, the multiple cognitive and emotional reasons and intentions of human agents means that it is not possible to predict with absolute certainty which one of a set of possible events will transpire. The social world is thus described as ‘open’ in contrast to the ‘closed system’ of a laboratory in which variables can be controlled and the impact of one variable on another repeatedly evidenced. Notwithstanding the relatively enduring nature of social relations, the social world remains intrinsically dynamic. Thus, one of the ways in which enquiry in the social world differs from enquiry in the physical world is in the treatment and explanation of the operation of mechanisms. In the social world, mechanisms can be considered as having a ‘tendency’ to operate in a particular manner (Collier, 1994; Bhaskar, 2008), but both their operation and the nature and extent of their impact will depend on the operation of multiple other mechanisms as well as human agency. This means that in the social world no constant conjunctions exist, events are not deductively predictable, and a counter example does not disprove the existence or operation of a mechanism (Collier, 1994; Bhaskar, 2008; Shipway, 2011). As Carter and New express it, human social life is “roughly patterned rather than law determined” (2004, p1). As a result, research in the open world must have explanatory and not predictive aims (Bhaskar, 1989; Carter and New, 2004).

In relation to the current study this perspective on the interplay between social relations and events impacts on the prominence afforded to the nature of the action and interaction between pupils. A critical realist analysis of interaction recognises the operation of structure and agency; rather than seeking to marginalise or view social concerns as context within which the mathematical learning occurs, it seeks to recognise their intrinsic role in it.

Critical realism neither prescribes nor proscribes specific research methods or approaches (Sayer, 2000; Carter and New, 2004), although its assumptions relating to the nature of social reality and meaningful enquiry necessarily guide the researcher’s selection. Indeed, Shipway (2011) suggests that a critical realist
perspective enables the researcher to appropriate existing research approaches and apply them with a more overt ontological grounding. Whilst Carter and New (2004) suggest that critical realism is ‘open ended’ in terms of the selection of research methodology, the relationship between critical realism and the methodology of enquiry is perhaps best described by Danermark’s phrase ‘critical pluralism’ (2002). The approach selected for this study is that of Education Design Research. The next section describes this approach and establishes its coherence with a critical realist perspective.

3.3 Education Design Research

The hypotheses and research questions set out in chapter 2 require a research approach that enable the exploration of the impact of pedagogy on pupil learning. Since the pedagogies in question were not already in place in the focus classrooms. I required a research design that extended beyond observation of current pedagogical practices and permitted the careful observation of the impact of adapted pedagogies on pupil learning. I selected Education Design Research, a relatively recent approach within education research (Anderson and Shattuck, 2012).

Whilst the diversity in both scope and structure of research within this field makes a precise definition challenging, education design research studies nevertheless share many common characteristics. Education design research is interventionist, theory oriented, operates iterative cycles, utilises real world classroom settings and employs collaborative partnerships between researchers and practitioners (Plomp and Nieveen, 2013; Barab and Squire, 2004; Cobb et al., 2003). McKenney and Reeves (2013) include a further characteristic that the research begins with an identified problem. Design research has similarities with, but also some differences from, action research. Both operate iterative cycles of intervention which may consist of new or altered teaching and learning approaches or activities, or organisational changes; both seek improvements to teaching and learning practice. However I selected design research for its greater focus on theory building (Anderson and Shattuck, 2012). Additionally, whilst both are conducted in classroom settings, action
research is usually conducted by practitioners, whereas design research is conducted by researchers in collaboration with practitioners (Anderson and Shattuck, 2012).

Van den Akker (2006) notes that the term design research can be seen as an umbrella term for several related approaches with differing goals, terminologies and definitions. Indeed as use of the approach grows, calls have been made to better articulate and define methodological guidelines in order to delineate the field occupied (Kelly, 2004). Gravemeijer and Cobb (2013) stress that design research should not be confused with quasi experimental studies which seek to evaluate the efficacy of one intervention over another. In a similar vein, the Design-Based Research Collective (DBRC) stress that design research seeks to do far more than designing and testing interventions. Authors point to research within the field which encompasses understanding of broader aspects of teaching and learning such as the development of social interactions, argumentation and teacher pedagogy (DBRC, 2003).

Characteristically the aims of design research are to develop theory that improves understanding of learning processes and to impact on practice (Barab and Squire, 2004; Gravemeijer and Cobb, 2013; Collins et al., 2004). However, whilst McKenney and Reeves (2013) identify practical application and theory building as simultaneous twin pursuits for all design research, Plomp and Nieveen (2013) identify these as two distinct branches of activity within the emergent field. The latter identify firstly development studies, or what might be termed ‘research based design’, which aim primarily, through the development and testing of interventions, to develop solutions to complex problems in educational practice. The second branch, validation studies or ‘design based research’ studies the processes of learning through the use of interventions with the primary aim of developing or validating theories about these processes (Plomp and Nieveen, 2013). It is the latter of the two, the validation studies, which align with my research aims. Whilst such studies do not necessarily meet the emancipatory agenda of critical realism, the ‘transformative agendas’ of researchers is nevertheless noted (Barab and Squire, 2004). Gravemeijer and Cobb (2013) however express caution regarding the use of the term ‘validation’ as a
descriptor of their type of design research; the term implies checking and does not sit well with the exploratory nature of either theirs, or indeed my own study.

Design research has its roots in pragmatism (Anderson and Shattuck, 2012; Barab and Squire, 2004; Cobb et al., 2003) and the discourse of empiricism is evident in some research reports, particularly those aiming to develop solutions to practical educational problems. However, the field of design research incorporates different branches of activity and a range of researcher positions. Cobb et al (2003) describe the classroom as a learning ecology, a complex interacting system, with elements of differing type and levels. This learning ecology includes the tasks and problems students address, the discourse practices, the norms of social and mathematical participation of the classroom. Cobb et al (2003; 2013) view design research as an opportunity to address this complexity and to use it to generate and test theory. In a similar vein, Shavelson et al (2003) suggest that design studies are multileveled and that events in a classroom should be viewed in connection to events and structures in the wider school or community. Kelly (2006) further stresses that design research does not support simplistic cause and effect models but seeks meaning making incorporating social, historical and cultural factors. Whilst these researchers do not overtly adopt a critical realist position, these indications of layered complexity and underpinning structures are not inconsistent with that of critical realism.

An ontological position is not defined so much by what you do, but by how you account for what you see. In essence, and in accordance with Shipway (2011), I am appropriating the structure of design based research and applying it with a more overt ontological grounding. In the next section, I outline my chosen research methods and detail how they are applied within the design research structure. In so doing, I establish their coherence with a critical realist perspective.

3.4 Study design

My research study focused on the mathematics learning of selected lower attaining pupils within the context of whole class lessons. Thus, it was in accordance with the key tenet of critical realism that educational research should take place in the ‘messy’
world of the classroom. In a similar vein, design research also stresses that interventions should be enacted in a holistic manner within the context of the classroom environment (van den Akker et al., 2006), recognising that the individuals within this classroom, their needs, goals and interactions will impact on how an intervention is enacted (Collins et al., 2004). For some design researchers the use and significance of this context reflects a typical pragmatic stance. In such instances characterising the classroom context is seen as important because this is the background against which events occur; understanding of such features enables differences in outcomes of an intervention to be understood or explained. From a critical realist stance these same features are indicative of the essential social relations of the classroom and are generative of and intrinsically involved in the events that occur. In the context of the current study the operation of social mechanisms, specifically authority relations, is viewed as contributing to the learning that emerges; these social relations are thus central to the study.

Common to both critical realism and my chosen branch within education design research is the selection of intensive research methods. Typically, these involve the detailed study of a small number of cases, involve close observation in the causal context, and employ qualitative analysis of the data generated. From a critical realist position, intensive approaches enable the researcher to move between observable phenomena and the mechanisms that generate them (Carter and New, 2004). These methods enable the researcher to address questions relating to ‘how’ a mechanism operates (Danermark, 2002; Sayer, 2000) and support the explanatory rather than predictive aim of critical realist research. The use of such methods and a focus on a small number of cases with an explanatory aim is one of several recognised approaches within education design research (Kelly, 2006; Gravemeijer and Cobb, 2013).

The open system of the classroom and the complexity of the interactions of mechanisms in each individual situation has implications for generalising findings and here coherence between design research and critical realist principles can again be established. Kelly (2006) argues that design research has a closer affinity with approaches that focus on a more ‘intimate’ definition of learning rather than those
aiming for an ‘averaged’ description. Indeed, design research does not attempt ‘context free’ generalisations, and recognise the difficulty in attempting replicability of findings (Barab and Squire, 2004; van den Akker et al., 2006); Cobb et al (2003) suggest that replicability is neither possible nor desirable. Instead they argue that the aim is ‘ecological validity’ such that the findings provide theoretical basis for adaptation to a new context. Thus, it is not so much the intervention outcomes that are to be generalised, but the theory underpinning the intervention. This is consistent with a critical realist view of generality; as Danermark (2002) notes, in critical realist research, generality is analytic rather than from a sample to a population. In the context of this study this same interpretation of generality applies; it is the operation of mechanisms, including the way in which the intervention impacts on their operation, that generality seeks to address.

Education design research incorporates three clear phases of activity (Plomp and Nieveen, 2013) the details of which vary depending on the type of design research under consideration. In my study, and following Gravemeijer & Cobb (2013; 2006), these three phases correspond to the initial planning and preparatory phase, the iterative trial and refinement of approaches and thirdly a phase of retrospective analysis and theory construction. The activities comprising these three phases is shown diagrammatically in Figure 3-1 overleaf and detailed in sections 3.5, 3.6 and 3.7 respectively.
Initial meeting with teachers; securing consent of all adults and children

School 1 Introductory visit
School 2 Introductory visit
School 3 Introductory visit

Visit to each teacher
Agree plans for Preliminary observation lesson and finalise focus pupil selection

School 1 Prelim Observation
School 2 Prelim Observation
School 3 Prelim Observation

Workshop meeting of research team

School 1 Research Lesson
School 2 Research Lesson
School 3 Research Lesson

Three cycles

Retrospective analysis and theory construction (Researcher only)

PHASE ONE

PHASE TWO

PHASE THREE

Development of conceptual framework and theoretical foundations of planned intervention

Figure 0.1 Research phases and activities
3.5 Phase 1: Preparation

Preparatory activities within this phase (Figure 3-1 above) serve to support the design and operation of class based trial. This begins with identification of the components of the conceptual framework and the exploration of these through the literature review (chapter 2). This enabled the development of the hypotheses and the formulation of the research questions (2.4). In line with the critical realist perspective of this research, this process enabled me to identify the contingent relation of the mixed attainment pair and to make conjectures about the operation of the social mechanisms involved (Danermark, 2002; Carter and New, 2004). These conjectured connections between the elements of my conceptual framework contributed directly to the development of the proposed intervention (Kwon et al., 2013).

In addition, the recruitment and selection of teachers and pupils was also part of this preparatory phase and this is detailed later in this chapter, in 3.8.1 and 3.8.2.

A further and significant element of the preparatory phase is the exposure of theoretically driven conjectures in the focus classroom contexts. This consisted of a preliminary observation of focus lower attainers and their higher attaining partners as they engaged in an agreed mathematics task. This preliminary lesson provided an opportunity to observe and analyse the way in which pupils worked together and to begin to understand how social relations impacted on the expression and use of mathematical awarenesses. In turn this supported the refining of the planned intervention. Preparing myself, recruiting and preparing the pupils and the teachers for this preliminary observation and for the subsequent research lessons required a sequence of activities as detailed in Table 3-1 below:
<table>
<thead>
<tr>
<th>Time frame</th>
<th>Activity and activity no.</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sep/Oct 2014</td>
<td>1. Meeting between researcher and each class teacher</td>
<td>a. Discuss patterns of participation and mathematical awareness of lower attaining pupils</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Teacher identification of possible focus pupils</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Secure consent from all parties</td>
</tr>
<tr>
<td>Oct 2014</td>
<td>2. Introductory visit: researcher joins class for a mathematics lesson</td>
<td>a. Meet pupils and discuss research study</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Acclimatise researcher, teacher and pupils to use of video equipment</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Explore technical use of cameras within the three class environments</td>
</tr>
<tr>
<td></td>
<td></td>
<td>d. Video and audio record activity during a mathematics lesson</td>
</tr>
<tr>
<td>Oct/Nov 2014</td>
<td>3. Second meeting with each class teacher</td>
<td>a. Finalise selection of focus pupils</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Tailor the agreed preliminary observation lesson outline</td>
</tr>
<tr>
<td></td>
<td></td>
<td>c. Engage in anticipatory thought experiment with teacher</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b. Video record mathematical activity of focus pupils</td>
</tr>
</tbody>
</table>

*Table 3-1 Timeline of school-based preparatory activities*

Arising from activity 2, each class teacher and I made decisions in relation to the positioning and use of video and audio equipment that enabled sufficient clarity of recording whilst not jeopardising pupil safety or teaching and learning activities. In addition, all activities within this preparatory process supported the development of relationships between each teacher and myself. The development and ongoing negotiation of these relationships is an essential aspect of my research methods (Maxwell, 2005; McKenney et al., 2006), and during this preparatory phase we began to establish our own norms of operation; how we interacted during the course of each lesson to evaluate its progress and to negotiate any alterations needed.

A field notebook was utilised in meetings to capture decisions made and questions raised. This same notebook was used during both classroom visits (activities 2 & 4) to note reflections on technical aspects of the use of the video cameras. Both classroom visits deepened my understanding of participation and interaction norms, pupil expression of mathematical awareness and teacher pedagogical practices. Together
these supported the development of the planned intervention. The second classroom visit (activity 4) was a focused observation of the mathematical activity of the focus pupils and the video and audio recordings of mathematical activity and interaction were subjected to detailed analysis. A synopsis of the activity used in this lesson is provided in 3.6.5.

Activity 3 above makes reference to ‘anticipatory thought experiments’. This phrase refers to the careful envisioning of the potential impact of the proposed intervention on pupils’ participation and learning and supports the research team to tailor intervention activities to most productively support the desired outcomes (Gravemeijer and Cobb, 2013). Such thought experiments, carried out prior to the preliminary and each of the research lessons, enabled refinements of the approach, for example in the layout and detail of recording sheets to enhance encouragement to notice and record in response to aspects of the task, and pupil developmental needs. Danermark (2002), describes such thought experiments as an important aspect of the retroductive inference (see 3.7) that enables connections to be made between mechanisms and observable events.

3.6 Phase 2: Iterative Intervention

Following the preliminary observation detailed above, the study moved into phase 2. This phase consisted of three cycles of planning, enacting and reviewing (Figure 3-1). Each cycle involved two main activities: a workshop meeting of the three teachers and myself, and the research lesson in each classroom which served to enact the intervention. In addition, reviewing data, finalising lesson activities and preparation of lesson resources was conducted as an ongoing task of the researcher. The subsections below detail the components comprising the intervention (3.6.1), the activities and organisation of the workshops (section 3.6.2), and research lesson organisation (section 3.6.3). Data collection methods across the preliminary and research lessons are detailed in section 3.6.4. Lastly, the mathematical activities for the preliminary and the three research lessons are detailed in 3.6.5.
3.6.1 The intervention: a pedagogical focus on noticing

The pedagogical focus on noticing is a key element of the study. My intention is not that each element of this pedagogical focus is examined separately for its impact. Rather, building on my knowledge of primary pedagogy, the focus comprises several elements which the teacher research team and myself considered would operate together to promote and sustain a focus on mathematical awareness throughout the lessons.

The elements of the pedagogical focus are as follows:

1. Use of tasks with strong affordances for noticing
2. Promotion of the enactive use of representations during paired activity
3. Teacher focus on ‘what do you notice?’ through class introduction, displayed speech bubbles, interventions at tables, and during class plenary discussions
4. Recording sheets which included large speech bubbles inviting the recording of noticing

The design of the intervention builds on literature examined in chapter 2. In all lessons including the preliminary lesson class teachers used an ‘open structured’ task (Boaler and Staples, 2008; Boaler, 2016). As noted in 2.1.4, the investigative or problem solving nature of these tasks affords opportunities for all pupils within mixed groups to engage. Additionally since their structure also encourages the development of reasoning and generality (Boaler, 2016), it follows that they will have strong affordances for noticing as a precursor to reasoning. An analysis of affordances for noticing of each task is included in appendix A.

The potential for the enactive use of representation to provoke noticing was considered in 2.2.4. In particular the support that such representation may provide to lower attainers to articulate relationships (Foote and Lambert, 2011) to notice generality (Houssart, 2004) and to support awareness of underlying mathematical structure (Mason and Johnston-Wilder, 2006) supports its inclusion as part of the intervention.
Lastly the high value given to noticing promoted through the teachers’ introductions, their interventions at tables, their encouragement to notice and the structure of the recording sheets is founded on the work of Voutsina and Ismael (2011), Hewitt (2001) and Mulligan (2011). All these authors promote an active focus on awareness in mathematics teaching and learning; whilst all components of the intervention comprise this active focus, the last two represent the most overt manifestations.

The use of the term ‘noticing’ rather than the term ‘awareness’ reflected our view that the former was a more accessible term for primary aged pupils. The repeated use of the term in teacher introduction, plenary and when talking to pupils at their tables was intended to maintain pupil attention on their own awareness of features and to promote the valuing of what each pupil noticed as they discussed the tasks together.

3.6.2 Workshops

Workshop sessions lasted for one hour and were held in the three teachers’ classrooms on a rotating basis. Each workshop session involved:

- Discussion of focus lower attaining pupils’ participation and their mathematical awareness in the context of the activity.
- The shared viewing and analysis of selected video extracts. Together with each teacher’s and my own direct experience of the lessons these extracts facilitated inference of the way the intervention impacted on the events observed.
- Discussion of the lesson content and teaching approach for the subsequent lesson in the light of the preceding discussion. This included the use of thought experiments (Danermark, 2002) and led to decisions about adaptation of the activities planned.
- Discussion of refinements to the intervention.

The video extracts used in these workshops were selected by myself. These extracts reflected both commonality and some exceptions in the awarenesses that lower attainers demonstrated and highlighted verbal exchanges between pupils connected
with these awarenesses. Over the course of the project each teacher/class was represented in the clips utilised for this shared analysis on several occasions. Use of shared analysis of video clips has been shown to support teachers to gain greater awareness of pupils’ mathematical thinking (van Es and Sherin, 2010) and in this research this shared analysis aimed to generate collaborative interpretations of mathematical action and interaction as well as to further develop research team relationships (Maxwell, 2005). The participating teachers offered insights into learning based on their experience and knowledge of the focus pupils that contributed significantly to our shared interpretation of the video extracts, to our understanding of the operation of mechanisms and to the further refinements of our approach. In the context of the small number of research lessons, refinements made were minor, reflecting task specific adaptations rather than adjustments to the approach. Thus, the intervention itself can be considered to remain constant over the course of the study and represents one design research cycle.

The aim of this study, and indeed of education design research in general, is not just to impact on learning in individual classrooms, but to generate theory relating to the processes of learning. Since it would be fallacious to assume that the same intervention can be enacted identically across different classrooms (Collins et al., 2004) adaptations of the intervention for different classroom situations are essential aspects of a coherent design. From both a design research and a critical realist perspective, such variability does not impact negatively on the potential for attribution of causality or theory construction (Maxwell, 2005; Gravemeijer and Cobb, 2013). Indeed, such embracing of complexity and variability is consistent with the critical realist drive to understand the operation of mechanisms in the open system of the social world. In practice such variability reflected differences between teachers in their styles of interaction with their pupils, for example in the way that the task and the focus on noticing was introduced and reflected differing pupil needs, at least partially in response to the different ages of the pupils concerned.

Whilst the workshop sessions were not a research focus in themselves, they contributed to the ongoing analysis, and were the forum in which decisions about refinements to the interventions were made. For this reason, I recorded notes on the
discussions using a field note book. Throughout the research period, ongoing emails constitute further records of the decisions made both collectively and in relation to class specific adaptations.

The final workshop session following the last research lesson additionally served as an evaluation meeting enabling discussion on the processes and outcomes of the project for teachers and pupils and to identify potential future avenues of enquiry.

3.6.3 Research lessons

Each research lesson was taught by the class teacher using the lesson activities jointly agreed at the preceding workshop. I had agreed with each teacher that I would prepare and supply necessary resources to support the activities of the lesson; this was in recognition of the need to consider my participants’ workload and to render their participation manageable (Maxwell, 2005).

Immediately prior to each lesson I discussed any last minute issues with the class teacher. This included discussion of any pupil absences that would change the pairings for the lesson and any social or learning issues that had arisen that would affect the way that the lesson was to be led or managed. In practice these discussions took place during a break time between lessons or before the beginning of the school day and needed to be achieved sensitively and with consideration for the existing demands on the class teacher’s time. During this time I was also engaged in setting up video and audio recording equipment.

During the lesson the class teacher was engaged in teaching and supporting pupil learning including for the focus pairs of pupils and retained responsibility for managing behaviour. This meant that the class teacher introduced the activity, was responsible for highlighting the focus on noticing, led discussion at pupils’ tables, and led lesson plenaries. I sought to establish what has been described as ‘balanced participation’ (Savin-Baden and Howell Major, 2013). This term reflects a researcher position mid-way between non- and full- participant observation and was in recognition of the need to sometimes participate in the activities of the classroom in order for my presence to be comfortable for pupils. In practice this meant that whilst
my primary responsibility was to manage data collection, I also engaged in dialogue with pupils in response to their questions. Over time, as I became an increasingly familiar figure in the classroom, pupils increasingly sought to discuss their work with me and ask questions. In engaging in such activity, which led me into the role of an additional teacher, I consistently mirrored the styles of questioning and intervention that as a research team we had agreed for the lesson.

During the lessons the class teacher and I communicated in a variety of ways to negotiate the progression of activities. Whilst ultimately the decisions in this regard remained within the class teacher’s domain, the collaborative nature of the research lessons meant that a combination of gesture and direct speech at different times served to signal agreement that an activity should either continue or come to an end. In addition, at opportune moments during the lesson, the class teacher and I exchanged our ongoing assessments of pupils’ progress and these served to guide the remainder of the lesson. Following each lesson, there was opportunity for a short debriefing session that supported immediate reflection.

### 3.6.4 Data collection activities and processes

Maxwell (2005) argues that there is no direct transformation of research questions into data collection methods; the latter needs to depend not only on the former but on the research situation and what will work most effectively. This was particularly pertinent when considering collection of data relating to pupils’ action and interaction, where decisions needed to pay strong heed to ethical considerations relating to their learning (see 3.8). The practical constraints of the school timetables and thus availability of both pupils and teachers also informed the decision to contain data collection activities solely within the lesson activity time.
In the preliminary observation and in the three research lessons the following data collection activities took place:

- video recording of the activity and interaction of two focus pairs of pupils in each classroom;
- additional audio recording of the teacher and all pupils during whole class interaction;
- photographic record of pupils’ inscriptions and diagrams; their findings recorded in books or on paper and of board annotations made by the teachers;
- field notes to reflect off camera discussions between the teacher and researcher, and occasional other off camera activity.

In addition, records of mathematical tasks and of all artefacts used to support teaching and learning have been collated.

Video capture was achieved using two free standing cameras attached to tall tripods. These were positioned close to the tables at which focus pairs of pupils were seated and thus each camera focused on one pair of pupils; this corresponds to the ‘stable mid-shot’ described by Luff & Heath (2012). The camera height and position enabled both the resources and artefacts on the tables, together with pupils’ faces, torsos and hand/arms to be within shot. The use of video is considered an effective means of capturing a frequently swift interplay of action and interaction (Heath et al., 2010; Derry et al., 2010). Such an interplay that includes speech, facial expression and gesture, resource manipulation and recording at table level is characteristic of primary school mathematics activity. Once focused and switched to record, the cameras did not need constant attention, however I monitored both frequently adjusting focus and direction to accommodate any shifting of the pupils’ position to ensure that recording quality and clarity was maintained.

The use of audio recorders supplemented the video recordings in two important ways. Firstly, these provided additional trace of the speech of the focus pair in the event that the microphone integrated into the video camera did not provide
sufficient clarity of speech in the context of a sometimes noisy classroom. Secondly, audio recorders captured the verbal interaction of the teacher and pupils during teacher led whole class sections of the lesson. During these times, the cameras remained trained on the focus pupils at their tables; these parts of the lesson involved the whole class engaging in a teacher led discussion, but frequently also involved short pair discussion or table top activity. In one class the teacher frequently asked pupils to gather on the carpet for such whole class activity and in these instances, again in order to follow the discussion of the focus pair, the camera followed these pupils.

The use of the field note book enabled me to note off camera activities and other contextual information to support later analysis of the video (Heath et al., 2010). This was not frequently needed but was occasionally useful when the teacher was orchestrating whole class activity. Here I used the notebook to record pedagogical moves on the part of the teacher that aimed to guide pupils’ focus of attention (Lobato et al., 2013) when these were not captured on video. Such additional notes in both the preliminary lesson and in all research lessons were in the form of unstructured observational field notes (Gillham, 2008) using a plain paper notebook with simple columns to record timings and observed behaviours.

From the preliminary observation and three research lessons in each of the three classrooms, with two cameras in operation in each class, a total of 24 video sequences of up to one hour were anticipated. This was reduced to 23 due to a camera failure in one classroom on one occasion.

3.6.5 Lesson activity synopses

The activities in which the pupils engaged in the preliminary and the research lessons are outlined below. A list of the affordances for key awarenesses in categories of mathematical pattern and property and mathematical structure provided by each activity is given in appendix A.
Preliminary lesson ‘The dice train’ task synopsis

This activity (Nrich, 2014), utilises three dice (shown by blue dice in figure 3-2) arranged as the carriages on a steam train with a fourth die (shown by white die in fig 3-2) acting as the funnel. There are two rules:

1. **All touching faces on adjacent dice must have the same number of spots.**

Here the hidden bottom face of the ‘funnel’ die is 3 and the top face of the carriage underneath is also 3. The touching faces of each pair of blue carriages are also the same each time.

2. **The number on the top of the ‘funnel’ (shown here as 4) must equal the total of the numbers showing on top of the remaining dice (carriages) that can be seen** (shown here as 2+2).

The spot total for an arrangement is found by summing the spots on all visible faces including those on elevations not visible in the picture above. The spot total for the train in the image is 55.

In the lesson introduction the teachers explained the rules and used a set of large foam dice to model the construction of a train conforming to rule 1. This constructed train was then evaluated against rule 2 and the challenge set was to construct trains that conformed to both rules. Teachers confirmed pupil understanding of the rules.
Pupils were asked to make as many trains fulfilling the rules as they could and to use the prepared recording sheet to record these trains and the spot total for each one.

Research Lesson 1 ‘Magic V’ task synopsis

The ‘Magic V’ task (Nrich, 2015a) requires pupils to use the numbers 1-5 and to arrange them into a V shape such that both ‘arms’ of the V sum to the same total. In Figure 3-3 the left hand image presents a Magic V since both ‘arms’ sum to 8 [1+4+3=8; 1+2+5=8]; the right hand image does not represent a Magic V since the two ‘arms’ sum to different amounts [2+5+1=8; 2+3+4=9].

![Figure 0-3 Magic V](image)

In the three research classrooms, pupils were presented with the image shown in fig 3-3 and were told that one of these V shapes was magic and that the other was not. They were invited to discuss the two V shaped images and to decide which was magic and why. Following a class discussion in which the rules of the activity were explained and the criteria of ‘magic’ clarified, pupils worked in pairs using digit cards and a prepared recording sheet to find solutions. They were encouraged to discuss what they noticed, what was and was not possible.

Subsequently, extensions to the task included using number cards 2 to 6 and identifying how this affected what was and was not possible, and then being given a target arm total and attempting to find the Magic V in which this total occurred.

Research lesson 2 ‘remainders after division’ task synopsis

The task for lesson 2 was devised by the research team. Pupils were given a set of number cards showing the numbers 1 to 20 and were challenged to make ten pairs of numbers from this set. The criterion for pairing two cards together was that when the number represented on one card was divided by the number represented on the
other, the quotient should be a whole number and there should be no remainder. Thus 10 could be paired with 2 because 10 divided by 2 gives a quotient of 5 and no remainder. 10 could not be paired with 3 because 10 divided by 3 gives a quotient of 3 and a remainder of 1. Pupils were advised to spread out their set of number cards in two rows of ten at the start of the task in order that all cards were visible. They were also advised to keep the selected pairs visible so that changes could be made as the task progressed. Pupils were asked to discuss what they noticed as they paired the cards and to decide whether or not the task was possible and to give a reason for this.

Subsequent extension to the task included pairing the same set of cards with a different criterion, such as that all pairs needed to divide with a remainder of 1 or pairing all cards such that the remainders from all pairs summed to the largest number possible. Variations on this extension were agreed with each teacher based on their knowledge of the development needs of the class.

Research lesson 3 ‘What numbers can we make?’ task synopsis

The starting point for the task ‘What numbers can we make?’ (Nrich, 2015b) was the image below (Figure 3-4) with the following instruction:

“Here are some bags containing lots of 1s, 4s, 7s and 10s. Choose any three numbers from the bags and add them together. What do you notice?”

Pupils were told that they could use numbers from any bag, that more than one number could be chosen from the same bag or that all three numbers could come from the same bag. They could choose from any bag as many times as they wanted – there was an unlimited supply of each of the numbers in each bag. Pupils were
advised to make several trials, recording each calculation and to discuss what they noticed. Following identification of the common property of the totals obtained, pupils were asked to explain this finding.

3.7 Phase 3: Analysis

Within an education design research structure, this phase of activity (Figure 3-1) concerns the retrospective analysis of the entire data set in order to generate or refine theories about learning (Gravemeijer and Cobb, 2013). Since this phase frequently leads to further investigations, Plomp and Nieveen (2013) prefer to describe this phase as ‘semi-summative’. Within this study, all analysis was underpinned by critical realist assumptions about the nature of the social world and the need for social research to deepen understanding of the operation of social mechanisms (Archer, 1998; Bhaskar, 1998). In this case this related to the authority relations arising from the mixed attainment pairing and the impact this might have on expression and use of mathematical awareness in lower attaining pupils.

Critical realism asserts that these structures and social relations may not be directly observable, rather it is manifestations of them in the behaviours and actions of individual agents that are the events observed (Danermark, 2002; Archer, 1998). Thus, analysis aims, through a retroductive process, to establish connections between individual events and the underpinning operation of social mechanisms. Danermark (2002) explains that retroductive inference is not a formalised mode of inference, but that it provides a means of moving from knowledge of one thing to knowledge of another. It is a transcendental form of argumentation by which one moves from observable events to their transfactual conditions, i.e. those conditions without which the events could not occur. Thus, retroduction seeks to answer the question ‘what must be the case for phenomenon X to occur and to occur as it does?’ (Sayer, 2010; Danermark, 2002; Lawson, 1998). Whilst different models of retroductive inference have been developed for different situations (Danermark, 2002; Collier, 1994), these different versions essentially follow a similar set of processes which begin with the observable events and move from these to a description of the operation of causal mechanisms.
Both Porpora (2015) and Danermark (2002) stress the need to start with, and place value on, description of what is frequently a complex event. This is my first stage. My second stage builds on the critical realist analysis of causality which identifies the ‘roughly patterned rather than law determined’ (Carter and New, 2004, p1) nature of the social world. In contrast to the constant conjunction view of causality in which event X is always followed by event Y, critical realists look for rough trends or ‘demi-regularities’ (Fletcher, 2016; Lawson, 1998; Danermark, 2002) in events observed in order to guide further analysis. In the current study this took the form of identifying sections across each video in which events that reflected change in lower attainers’ activity and from which the development of lower attainers’ awarenesses might be inferred. These ‘awareness events’ were then collated across lessons and focus pupils to identify commonality and differences. The third stage of my analysis involves abduction, the theoretical interpretation of these demi-regularities (Danermark, 2002), in this case the awareness events. This abductive process provides new insights into the events by explaining them through a theoretical lens, deepening understanding of them, whilst testing out the application of theory in so doing. It must however be acknowledged that such explanations and the selection of theory are necessarily fallible (Danermark, 2002; Olsen, 2004; Sayer, 2010; Fletcher, 2016).

The fourth stage builds closely on the third in identifying and justifying through reasoning with the empirical data the causal mechanisms producing the events observed. Here, the use of different examples from data, together with theory enable the researcher to establish the tendencies of the mechanism to operate in the situations studied. This stage incorporates further development of theory of the processes involved (Fletcher, 2016; Danermark, 2002).

The following two sections provide more detail in relation to the activities undertaken in stages 1 and 2 respectively. These stages focused on the extraction of data from the video sequences and the use of this to identify demi-regularities or commonalities. Section 3.7.3 provides summary tables of the analytic frameworks used in stage 3 to identify firstly awarenesses and secondly features reflecting authority relations between the pupil pair. Stage 3 is illustrated through the use of
literature to interpret and explain events in chapters 4 and 5; and stage 4 is exemplified through the discussion of the operation of mechanisms in chapter 6.

3.7.1 Analysis Stage 1: Description

Fitzgerald, Hackling and Dawson (2013), note that video data enables the researcher to attend to the ‘layers of complexity that are inherent in the acts of teaching and learning’ (p58). Its permanence means that recorded events can be subjected to multiple viewings from different perspectives and in flexible ways with opportunity for freeze frame, silent, slow, real and fast time motion in order to focus on and examine particular features or events (Powell et al., 2003; Derry et al., 2010). In creating a detailed post hoc description, the video data provided a rich source that no other data collection method would have permitted.

However, these opportunities also present challenges, chief amongst which is the need to maintain focus and to tease out meaning from these layers of complexity. Rostvall and West (2005) describe the technological capacity provided by video as ‘deceptive’ in the ease with which action and interaction can be captured; the analysis of this dense flow of information presenting challenges in being both transparent and systematic. The sheer quantity of data together with the complexity it presents lead Luff and Heath (2012) to reflect that video data can seem ‘strangely resistant to analysis’ and that it is ‘difficult...to break it apart into fragments that can be subjected to analysis’ (p258).

The construction of a description of the events and actions contained in the video sequence is the first stage in teasing out meaning. This approach falls within an inductive approach to video analysis in that I am subjecting the data to progressively finer grained analysis. In contrast, a deductive approach would involve systematically sampling across the entire data set (Derry et al., 2010).

Recording a description inevitably reflects judgments made in terms of what is and is not included in such a description. An initial, or several viewings of the entire video prior to recording a description enables constructs to emerge rather than being imposed and enables the researcher to gain an understanding of the entirety of the
sequence before it is subjected overtly and explicitly to an analytic lens (Fitzgerald et al., 2013; Powell et al., 2003; Erickson, 2006). Alongside such uninterrupted viewing, Erickson (2006), suggests that the researcher makes the equivalent of field notes, or analytic memos (Powell et al., 2003) to chart potential lines of enquiry.

My initial intention was to watch each video and to make brief analytic memoranda as I did so, noting timings as accurately as I could but without frequent pausing of the action. Since I had been present in the lesson and was familiar with the events, I considered this dual watching and note taking would be manageable. In practice I found this not to be the case; the making of notes was not sufficiently focused despite my guiding questions and the process of analytic memoing interfered with my capacity to watch the tape closely. I found I missed events and wanted to replay sections; this then interfered with the sense of progression in the lesson as a whole. As a result I opted just to watch the video in its entirety without any note taking or noting timings, making mental notes only. I recognised that in order to make this first uninterrupted viewing productive it was vital that it was closely followed by the construction of the detailed description that is the focus of this stage. I ensured that both could take place at the same sitting.

The construction of the detailed description necessitated the video being separated into timed sections (Powell et al., 2003; Erickson, 2006). In practice the separation of content into timed chunks was relatively straightforward. These were not equal length intervals but thematic, activity driven sections; each comprised a period of time in which a particular activity was underway. When activity changed, or was interrupted, a new section began. Thus, a teacher approaching the table to talk, a pupil leaving the table, a mini plenary, a change in the nature or focus of pupils’ activity all stimulated the separation of one section from another. A section might be up to five minutes long, but the typical duration was 2-3 minutes. As I recorded descriptions, I ensured that these were not inferential, judgmental or interpretative (Mason, 2011; Powell et al., 2003) but that they charted events and actions and included reported content of speech. I utilised an electronic spreadsheet for this activity (see sample in appendix B).
The completion of this description immediately after uninterrupted watching of the entire sequence meant that my memory of the overall shape and content of the lesson was strong; I had mentally logged many of the transition points and events and begun to notice some patterns. Significantly, the compilation of the detailed descriptions brought a much deeper and more detailed familiarity with the events and each individual’s part in them. At times this challenged the view I had gained from the initial uninterrupted viewing, for example in relation to which of the pair of pupils had spoken first or acted in a particular sequence of activity. Thus this was far more than a mechanical logging of the video content. It was during this phase that questions and hypotheses about meaning, relevance and interpretation were raised and noted, where enthusiasm and uncertainty rose and fell and where a priori and post priori research questions (Powell et al., 2003) were refined and refocused. This was a significant part of the analytic process and whilst time consuming, I consider it time well spent.

3.7.2 Analysis Stage 2: Identification and collation of awareness events

In line with the critical realist approach of beginning with the observable events, this stage of analysis focused on identifying ‘rough patterns’ (Fletcher, 2016; Lawson, 1998) in the data. Since the patterns that I was interested in related to the demonstration of mathematical awarenesses, and the nature of accompanying interaction, this phase began with the identification of video sections which might provide insights into these features. Following Powell et al (2003) I have termed these ‘critical sequences’. The selection of these critical sequences is made on the basis of their relevance to the research questions and involves an ‘explicit act of interpretation’ (Knoblauch and Schnettler, 2012, p336). In making such a selection from video data other researchers have used this stage of analysis to either identify representative examples of events (Borko et al., 2008), to chart the range of different events or activities (McNaughton, 2009), or to identify events marking change from what has occurred before (Powell et al., 2003). In line with my research questions I identified critical sequences which reflected changes in the lower attainer’s mathematical activity from which new awareness might be inferred.
The first critical sequence of each video was the first section of pair activity in which both pupils demonstrated awareness of the rules of the activity. I identified this video section through observation of their activity and using evidence from speech. In this first sequence I noted features of their mathematical activity (see Table 3-2 in section 3.7.3 below) which permitted inference of early or pre-existing awarenesses. I also focused attention on the following features of this early task work: how decisions were made, by whom, who instigated activity, how each pupil was involved in the activity underway in the video section, what was recorded, by whom. This focus on the early parts of pair activity enabled me to characterise pair interaction patterns and this provided a basis for identifying changes in the nature of interaction across a lesson and between lessons.

Beyond this initial sequence the selection of additional critical sequences built on identification of changed or new aspects of the lower attainer’s mathematical activity. This drew on three features of pupils’ interaction and activity. Firstly, I examined their direct speech to identify new reference to mathematical facts, connections, patterns, or properties. Secondly, I drew on their use of resources in constructing mathematical relationships, such as the use of Numicon to identify a number as a factor of another, even if an utterance of this number relationship was not made. Thirdly, I drew on gesture and markings made which indicated that a pupil had noted a relevant feature (such as pointing to a recorded feature or resource construction) I particularly focused on the speech, gesture and actions of the lower attaining pupil, but where I could identify that an utterance from the higher attaining pupil was built upon by their partner this was also noted. Sections selected represent change or reflected that a previously noted feature was built upon in some way. Thus, for example, the new noting that a number was divisible by 2 would be reflected in the selection of the video section where this was first observed. However, repeated identification by a pupil of this same feature would not result in repeated selection of sections where this was evidenced unless the pupil was using this fact in a new way, for example to reason about this finding or make a conjecture based upon it.

This was a careful and reflexive process. During stage 1, as I constructed the description of events, I had used the cell shading function of the electronic
spreadsheet to highlight cells relating to video sections that were ‘potentially of interest’ because of the nature of the mathematical activity in which the pupils were engaged. In stage 2, I returned to these and deliberately questioned my own interest in these sections to support the development of my rationale for the final selection. This involved constant interrogation of how I could justify that a change in activity permitted inference of mathematical awareness. Thus, decisions emerged gradually and required an iterative process of revisiting sequences and considering additional sequences to re-affirm decisions to select.

All critical sequences involved child pair activity exclusively. Sections where the teacher or another adult was in conversation with one or both pupils were marked for easy later retrieval should they be required to support later explanation or interpretation of events and processes.

This process resulted in the selection of 5-7 critical sequences of between 50 seconds and 3 minutes from each full video. The next step required me to scrutinise these sequences in detail in order that specific utterances or actions that enabled inference of awareness could be identified and then that these could be collated across focus pupils and across lessons in order to identify commonality and differences. These same sequences were also examined in relation to the second analytic focus of authority relations. Categories and examples of action and interaction that supported this second analysis focus are given in Table 3-3 section 3.7.3 below.

Subsequent to the identification of critical sequences, Erickson (2006) advocates the production of detailed transcriptions of both the verbal and non-verbal actions in the selected tape sequences in order to support discourse and conversational analysis. Powell et al (2003) suggest a similar but less detailed next step involving the construction of a narrative summary of the activity within the critical sequence, a transcript of its dialogue and a summary of immediately preceding activity to provide context.

Key to decisions regarding the next steps in analysing video data is consideration of how transcripts can further illuminate rather than replace data (Heath et al., 2010).
Whilst the production and use of transcripts is common practice, Derry et al (2010) note that no transcript can capture the complexity of events. The production of a transcript requires judgements to be made about what is and is not included; such judgements reflecting implicit or explicit bias and value judgements. In short no transcript is neutral or objective (Derry et al., 2010; Powell et al., 2003; Erickson, 2006; Rostvall and West, 2005). Nevertheless, Powell et al (2003) identify that transcribing enables extended consideration of dialogue, that it can reveal mathematical meaning, and that it illuminates features not discerned through the viewing of the video directly.

Drawing on the above considerations I opted to transcribe the critical sequences and to include descriptions of activity within these transcripts. I also included summary notes relating to the nature of the interaction between the pupils. However, I wished to stay close to the video as data. Thus, the use of these transcripts to identify specific utterances or actions that provided evidence of a new awareness or to identify an authority feature was supported by repeated viewing of the critical sequences. This repeated viewing provided greater security that my interpretations were robust. An example of critical sequence transcript is shown in appendix C.

3.7.3 Analytic frameworks

Building on the literature examined in chapter 2, I identified four different awareness types (see 2.2.6). These are detailed in Table 3-2 overleaf, together with examples of action and interaction observed that supported inference of the awareness type in question. This analytic framework was used in analysis relating to research question 1.
Table 3-2 Analytic framework for mathematical awarenesses

<table>
<thead>
<tr>
<th>Code</th>
<th>Awareness category</th>
<th>Examples of speech or action</th>
</tr>
</thead>
</table>
| RA   | Awareness of the rules of the activity | Actions repeatedly in line with the rules of the task  
Assertion of rules as part of correcting a partner  
Use of, or articulation of rules as part of checking and confirming a solution |
| MP   | Awareness of mathematical processes | Observed use of mathematical working processes such as fixing a variable, systematically specialising, making lists of results  
Pupil’s assertion of a way to proceed or next step to take  
Observed use of calculation approaches |
| OP   | Awareness of mathematical patterns and properties | Noting through speech, gesture or inscription a feature of a result or outcome obtained  
Expressed intention to use of a feature of outcomes obtained to target a subsequent trial |
| S    | Awareness of structure or generality  
Subdivided into:  
a) Task/resource specific structure or generality  
b) Broader mathematical structure or generality | S:Tr (Task specific):  
Noting a relationship between the constraints of the task and the outcomes possible  
S:Ma (Broader mathematical):  
Noting a relationship between sets of numbers  
Noting a feature of a way a set of numbers is structured or organised  
Forming or challenging a conjecture or generalisation |

My second analytic focus of authority and influence required examination of the same critical sequences to explore how and in what ways the development of awareness was associated with patterns in pupil action and interaction. As noted in chapter 2, and drawing on literature examined in this chapter, analysis focused on both verbal interaction and pupil activity (see 2.3.5). Lower attainers’ articulation of mathematical awarenesses identified in line with the framework above (Table 3-2) were reinterpreted as claims to mathematical expertise. Analysis addressed the way in which these were responded to in speech and action by higher attaining partners. The components of this area of focus together with examples in each category are detailed in Table 3-3 overleaf:
The perceived intellectual merit of an idea (claims to mathematics expertise authority)

Analysis focuses first on the making of a claim, or the proposal of an idea
- Expression of mathematical awareness in the form of an idea, pattern, connection, conjecture, generality,

Response to the claim or idea
- Verbal response (assent or dissent) to the idea
- Action in line with either assent or dissent
- Questioning or checking of the accuracy of partner’s solution

Access to the activity space

Consideration of the lower attainer’s visual and physical access to resources
- Visual sight line of resource and activity underway
- Opportunity to act on resource without interruption

The degree of spatial privilege afforded

Extent to which lower attainer is physically attended to when speaking or acting
- Eye contact with lower attainer
- Visual attention paid to lower attainers activity
- Bodily orientation of higher attainer towards either lower attainer or the activity underway

Table 3-3 Analytic framework for authority and influence

As noted above, for both these analytic focus areas the same critical sequences were examined. However, where this left questions unanswered, earlier video sequences were occasionally re-examined to support interpretation of how patterns in interaction emerged or changed.

In order to respond to the fourth research question and in evaluating teachers’ intentions regarding the distribution of disciplinary authority (Grootenboer and Jorgensen, 2009), I draw on the three elements identified by Gresalfi and Cobb (2006) to enable consideration of the expectations, entitlements and obligations that the class teachers placed on pupils in introducing the tasks and intervening at tables. I interpret entitlements as opportunities that a pupil can take in making decisions and offering suggestions together with the behaviours that they should expect from their partner. Expectations are the requirements for activity that the teacher sets in terms of the goals of the task and what pupils are expected to focus on. I interpret a
pupil’s obligations as what they can be held accountable for, whether this be for the accuracy of their findings, the learning of their partner or their mode of recording.

3.8 Ethical considerations

In seeking new insights into learning processes through adjusting and reflecting on pedagogical changes, this research project falls within the routine activities of a classroom teacher. However, its deliberate application of an intervention together with its data collection activities required that I consider carefully the potential impact of the research processes on the participating teachers and pupils.

In the following sections I outline the provision made within the research design to ensure that the research was carried out in the most ‘conscientious and responsible manner possible’ (SoE, 2011, p1) and that it adhered to the principles outlined in BERA’s guidelines for educational research (BERA, 2011). These provisions led me to seek and gain approval at Tier 1 as specified in the School of Education Tier 1 protocol (SoE, 2011).

3.8.1 Recruitment and consent: schools and teacher participants

Participation in this study required teachers to open the mathematics teaching and learning in their classrooms to a mathematics education specialist and to engage, with a group of colleagues, in analysis of their own pedagogy and its impact on classroom learning. The characteristic limited confidence of primary class teachers in their own mathematics subject knowledge and teaching is a finding of much research (e.g. Rowland, 2009; Haylock, 2006) and I sought to work with teachers who displayed both confidence and enjoyment in the subject. The relatively transfactual nature of social relations (Archer et al., 1998) means that I had no reason to suppose that such greater confidence would mean that different authority relations would operate in these classrooms. However, I did conjecture that a greater confidence in their own mathematical subject and pedagogical understanding would render participants more open to the activities in which I wished them to engage. In particular I sought teachers who had already shown willingness to engage in reflexive analysis of their own practice with colleagues and thus approached graduates of the
Primary Mathematics Specialist Teacher Programme (MaST). At the time this was a DfE supported and funded programme led by the University of Brighton, aiming to develop the mathematics subject and pedagogical expertise of primary class teachers in England (Williams, 2008). Its participants were experienced primary class teachers who already displayed confidence and enthusiasm in their mathematics teaching; the course sought to deepen this knowledge and required them to engage in reflexive analysis of their practice as part of the course of study.

Drawing on former, successful participants of this programme ensured that my current role as tutor on the MaST programme was not a potential source of conflict in terms of an ongoing academic relationship. However it was important that I recognised that my previous role as their mathematics education tutor might influence mathematical discussions between us. I was cautious in all our workshop meetings to respect and draw upon their more contemporary knowledge of the detail of age related activity as we planned activities and to draw on their knowledge of the pupils to support analysis of video clips. I also remained aware of how my role as their former academic tutor enabled privileged access to this group of teachers. I was thus careful to avoid any impression of pressure to participate. I made initial contact via email with the entire cohort of former students inviting expression of interest without commitment on either part. This initial email identified the research focus and outlined the research methods involved. I clarified the need for mathematics teaching in the school to be organised in a standard mixed attainment class rather than a streamed or set class structure within the school. Those expressing interest were then sent an information sheet (appendix D); I followed this with a short visit to each of these teachers to discuss the project more fully.

At the end of this process, four teachers had expressed willingness to participate and I narrowed this down to three on the basis of the following criteria:

- Age range taught: I sought three classes with a similar age profile in order that the research lesson could address the same core lesson content. A wide span of age ranges would render this more challenging.
• Geographical proximity: to facilitate workshop sessions schools needed to be situated such that travel time was manageable for the teachers.

The one teacher who was not selected was contacted by email and by phone to acknowledge my gratitude for their interest and to outline the reason for not being selected in relation to the criteria above. The three teachers selected had between 3 and 14 years teaching experience at the beginning of the study.

Informed consent was sought from all adults who would be present in the classrooms during the research. As no teaching assistants were due to be working in the classrooms during the research, this only involved the three participating teachers. In accordance with BERA (2011) the information sheet (Appendix D) outlined the purpose and activities of the research, the confidential storing of and access to data and the right to withdraw. Building on this information sheet and our further discussions, all teachers gave their consent, and this was confirmed through signing of the consent form (Appendix E).

Prior to seeking the informed consent of those who had expressed initial interest, I secured permission to access these teachers and their classes with the relevant gatekeepers. I arranged a meeting with the head teachers of each school to discuss the purpose and structure of the study and to outline the risks and benefits (section 3.8.4) of participating for the adults and pupils concerned. Whilst the initial contact to generate teacher participants had already been made directly with teachers, it was only once the head teacher had agreed that consent from the teachers and their classes was sought.

My access to these teachers and schools arose through my professional role in the School of Education and I had a responsibility to maintain the reputation of the School of Education in its partnership work with local schools. Thus I informed both the Head of School and the Head of Partnership at the point of selection of schools and teachers in the event that either of these senior leaders had any concerns about my choice of schools in which to conduct this research.
3.8.2 Recruitment and consent: focus pupils

The class teacher made the initial selection of focus pupils. This selection was based on the class attainment profile, drawing on numerical summative attainment data from the end of the previous academic year (summer 2014) in addition to the current (autumn 2014) teacher assessments. In each class I sought two focus lower attaining pupils who would work with higher attaining partners. Whilst it was desirable that over the course of the study the pairings would remain constant, discussions with the class teachers revealed that there may be reasons, for example behavioural concerns, or pupil illness and absence, why this would be impractical, and I thus did not stipulate this as a requirement. As has been discussed in 2.1.1 ‘higher attaining’ and ‘lower attaining’ are relative terms; one can be lower attaining relative to national age-related expectations of attainment, and/or relative to others within the class. Since this study focused on mixed attainment classes within mainstream primary schools, the term lower attaining satisfied both these relative comparisons. Thus, in each class, two focus pupils operating at lower than national age-related expectations were each paired with a pupil either meeting or exceeding age-related expectations. The class teacher’s knowledge of the pupils together with the evidence gained by us both during the introductory visit enabled a joint final decision that built also on pupils’ response to the use of the video camera. During this visit, as different camera positions were explored I spoke to pupils about how it felt to be on camera and those appearing uncomfortable or expressing a clear wish not to be so were not selected.

Particular attention was paid to gaining informed consent from the pupils and their parents. All parents/carers were given an information sheet and consent form (appendix F). The content and style used on this form was agreed in conjunction with each class teacher and their head teacher to ensure it was appropriately worded. Both the class teacher’s and my name and contact details were given on this sheet and in agreement with the class teacher I made myself available either directly at the school or via the class teacher to answer any questions. In each of two classes, one pupil requested through the consent form not to be part of video capture; positioning of the video cameras ensured that these pupils were not in shot at any time.
Whilst analysis in this study has focused on lower attaining pupils, it is important to note that both higher and lower attaining pupils were key to the research. However, reference to lower/higher attainment in information given to parents was potentially problematic. I sought to achieve both openness (BERA, 2011, p6) and sensitivity. However, it was not possible to establish the extent to which a pupil’s current mathematical attainment either in relation to the class or to national expectations was understood by a parent or child. Thus it was particularly important that the selection, or non-selection, of a particular pupil to be part of the focus group, and thus part of video data capture, was not seen as either a judgement on, or affirmation of, their mathematical ability or potential. Such public interpretations of mathematical competence, should they be made, could impact on how pupils feel (Reay and Wiliam, 1999) and how they behave towards each other (Marks, 2013). This was particularly pertinent given my deliberate use of the phrase ‘lower attaining’ rather than the phrase ‘lower ability’; I wished to avoid and dispel the association with fixed potential that the latter phrase frequently conveys (Marks, 2013). Given the risk that the identification of ‘higher’ and ‘lower’ attainment would have a negative impact on pupils, and since pupils from across the attainment range could all be selected for the study, either as focus pupils or as their partners, there was no specific mention of lower or higher attainment in this information sheet for parents.

Homan (2001) cautions against the giving of consent by those who are not themselves the subject of the research; this is echoed in Heath et al (2010) who argue that explicit permission to video record should be sought from pupils and that the purposes and activities of the research should be explained in language that pupils can understand. Thus, as noted above, during the introductory visit I deliberately spoke to pupils about the research and what I and they would be doing in the research lessons. I discussed why I wanted to use the video and audio recorders and how they would be set up. I sought explicit verbal consent from pupils to record them at their work. Piper and Simons (2005) advise a ‘rolling informed consent’ the need for which arises from the difficulty of participants understanding the implications of participating in the research at the outset of the process. This meant that I continually sought explicit permission to record focus pupils during each research lesson.
3.8.3 Data security and participant anonymity

Digital audio and video data was securely stored on a password-protected computer, with an additional back up copy on an encrypted portable hard drive. The short video sections of classroom activity used in the workshops were transferred to a memory stick for the purposes of transporting them to the workshop location and were then deleted from this memory stick following the workshop meeting. My fieldwork journal did not name schools, teachers or pupils and was kept securely at my home when not in use in either a research lesson or a workshop session.

The information sheet for parents (appendix F) outlined the use of video sequences in the teacher workshop sessions. I explained that the video sequences were only viewed by the teachers participating in the study, that no additional copies were made and that these short sequences focused solely on particular instances of mathematical activity that enabled insights into pupils’ mathematical learning. The information sheet further explained that the audio and video data was to be used in this research only. Subsequent to the completion of the research lessons, a further consent form seeking permission to use the video data for teaching purposes and for research dissemination at conferences was sought. The majority of parents consented to this additional use; all audio and video data relating to children whose parents did not provide this additional consent will be deleted once the research process is complete.

In writing arising from this research for thesis, conference and publication, the identity of all participating schools, teachers and pupils is protected through the use of pseudonyms and descriptions of the schools that refer only to the age range of the pupils at the school and descriptors such as large, urban.

The information sheets confirmed that teachers and pupils had the right to withdraw at any point. It explained that should a pupil exercise this right then all data collected from this pupil that had not been used for analysis would be removed and destroyed. Further recordings made in the class would not focus on this pupil. The corresponding form for the teachers confirmed that should any participating teacher decide to
withdraw then any data gathered from this teacher that had not been used in analysis would be removed.

3.8.4 The risk of harm

This research project was designed to impact positively on pupils’ learning: the intervention was rooted in theory and research, and the ongoing reflection on pedagogy enabled through the workshops offered the potential for pedagogical developments that could benefit all pupils in the class. However, I carefully considered the impact of participation on those involved and in particular the potential disruption to pupil learning arising from my presence in the classroom, from the specific data collection activities, and to the increase in teacher workload.

I sought to establish a positive relationship with pupils through my openness in relation to the research purpose and my explanation of the use of video and audio recording. My familiarity with primary aged children and my balanced participation role in classrooms led me to be confident that I could interact with them in a positive and productive way such that my presence was not a distraction. This confidence was borne out through the research lessons where pupils increasingly and evidently welcomed my presence, seeking to engage me in task related conversation and to share their work and ideas with me.

Whilst any recording device has the potential to distract and distort activity, concern is more frequently raised in this regard in relation to the use of video rather than audio (Lankshear and Knobel, 2004) and for this reason the introductory visit was of prime importance. However, Heath et al (2010) advise that whilst the presence of a video camera is of occasional interest or distraction to research participants and may be commented on at times, it is also the case that where there is a task to be completed as in the case of a classroom, the demands of the task are prominent. This was borne out in this project. Pupils engaged in their tasks, rarely commenting on presence of the camera or desk microphone. Usually this occurred when one of the pair engaged in off task behaviour; indeed it might be argued that the occurrence of such occasional off task behaviour signalled a degree of comfort with the camera.
It is undeniable that participation in this project carried an additional workload for teachers. Whilst I provided support for planning research lessons, additional preparation time for each teacher was still inevitably involved. Attendance at workshop sessions and ongoing email contact additionally required a time commitment. Whilst I hoped that participation in the project generated enthusiasm and interest, discussions about workload formed part of the continuous negotiation of informed consent with the teacher group. All were able to maintain participation throughout the project, although one was not able to attend one of the workshops due to other work commitments. I visited this teacher separately to share the outcomes of the workshop missed and to discuss the following research lesson.

3.9 Chapter summary

This chapter has set out the processes involved at each stage of the research study and has demonstrated coherence between my ontological perspective and the research approach selected. The decisions made and the relationships established with the participating teachers and pupils enabled the study to be successfully conducted in the classroom; similarly the approach to analysis enabled a large amount of video material to be condensed and analysed in a productive manner. However, the decisions made gave rise to some limitations and these are addressed in chapter 7.

This chapter has detailed the approach to analysis and this paves the way for the next chapter to focus on the outcomes of this analysis.
Chapter 4: The impact of the focus on noticing on lower attaining pupils’ mathematical awareness

The two analysis chapters (chapters 4 and 5) present findings and analysis in relation to the four research questions. Chapter 4 addresses the impact of the focus on noticing on the nature and impact of lower attaining pupils’ mathematical awareness (research questions 1 and 2). Chapter 5 addresses the impact of the focus on noticing on the way that disciplinary authority was distributed and on the nature of interaction between the pair (research questions 3 and 4) and considers how this contributed to mathematical learning and progress in the task.

The following summarises the findings associated with research question 1 and 2:

**Research question 1:** ‘In what ways does a pedagogical focus on noticing impact on the nature of lower attaining pupils’ mathematical awarenesses?’

a) The focus on noticing was associated with lower attaining pupils frequently demonstrating awareness of structure and generality

b) The focus on noticing had limited impact on lower attaining pupils’ awareness of the mathematical patterns and properties of their findings and results.

**Research question 2:** ‘How do lower attainers’ mathematical awarenesses contribute to task progress for the pupil pair?’

c) Lower attainers’ expressed awarenesses of structure and generality made significant contribution to task progress.

This chapter first presents analysis relating to awareness of mathematical structure and thus addresses findings (a) and (c). This is followed, in section 4.2, by analysis relating to awareness of mathematical pattern and property and considers finding (b) above.

In line with the methodology of Education Design Research, the preliminary lesson was part of the preparatory phase of the study; its outcomes contributed to the design of the intervention and to the focusing of the research questions. However, the presentation and analysis of data from the preliminary lesson is positioned after
consideration of data from the research lessons in each relevant section. This enables the comparison between the development of awarenesses to be better illustrated.

Data was collected from two focus lower attainers and their partners across three classrooms and four lessons. Early scrutiny of data showed that some video sequences should be removed from the data set. This applied for example where the video or audio recording was of poor quality or work was frequently interrupted by other pupils. Two further sequences were removed because the focus pair independently manipulated the constraints of the task in such a way that the affordances for noticing were substantially reduced. Where removing such sequences resulted in fewer than three available video sequences from any focus lower attainer, this lower attainer was removed from the data set completely.

This process resulted in a total of 13 video sequences from the original 23 being subjected to detailed scrutiny. The original six focus lower attainers were reduced to four through this process. This resultant data set is summarised in the following table:

<table>
<thead>
<tr>
<th>Lower attainer</th>
<th>Higher attaining partner</th>
<th>Data available from:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rose</td>
<td>Poppy (preliminary lesson)</td>
<td>Preliminary lesson,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Research lessons 2 and 3</td>
</tr>
<tr>
<td>Joe</td>
<td>Ash</td>
<td>Preliminary lesson,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Research lessons 1 and 3</td>
</tr>
<tr>
<td>Mia</td>
<td>Ruby</td>
<td>All lessons</td>
</tr>
<tr>
<td>Daisy</td>
<td>Alice (preliminary lesson)</td>
<td>Preliminary lesson,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Research lessons 1 and 3</td>
</tr>
<tr>
<td></td>
<td>Jess (research lessons)</td>
<td></td>
</tr>
</tbody>
</table>

*Table 4-1 Data availability*

As noted in 3.8.2 whilst consistency of focus pupils was desirable, it could not be guaranteed; as Table 4-1 shows, over the course of the study all focus lower attainers remained constant but for reasons including absence of particular pupils, on occasion their higher attaining partners did change. Data is available for four lower attainers for the preliminary lesson, for three lower attainers for research lesson 1; two for research lesson 2 and all four for research lesson 3.
4.1 Awareness of structure and generality

This section addresses findings (a) and (c), that

a) A focus on noticing was associated with lower attaining pupils frequently demonstrating awareness of structure and generality

b) Lower attainers’ expressed awarenesses of structure and generality made significant contribution to task progress.

Across the three research lessons and the preliminary lesson important structural awarenesses were demonstrated by lower attaining pupils. The sub-sections below address the development and use of awareness of structure and generality in each of the research lessons followed by discussion of use of awareness of structure and generality in the preliminary lesson. Each sub-section includes an awareness pathway diagram which represents the development of awarenesses; implications arising from comparison of these pathways is further considered in 4.1.5.

4.1.1 Research lesson 1

Two of the three lower attainers for whom data from research lesson 1 (RL1) is available, developed and demonstrated a significant task related mathematical structural awareness during the course of their activity. The third lower attainer, Daisy, did not independently articulate or demonstrate this awareness through her action. She did however report understanding of this same idea when it was explained to her by her partner. No structural awareness is reported for Daisy in this lesson.
For both lower attainers Mia and Joe, separately demonstrating this awareness, progress reflected a development from the partial structural awareness (Mulligan and Mitchelmore, 2009) which had been evident in the first critical sequence for each pair. This partial structural awareness took the form of treating the V shape as being comprised of two lines, each of which contained 3 numbers, (shown as a left and right arm each having three orange discs in figure 4-1). Evidence for this inference, together with the development from this point for both pupils is detailed below.

For lower attainer Joe, working with higher attainer Ash, progression from this point was particularly marked.

Joe began the task using a random specialising approach (Mason et al., 2010) in which cards were switched around swiftly with no stability of any position or number drawn on to support systematic exploration; attention was not productively focused (Owens, 2004). Joe’s awareness of the rules of the activity in this first critical sequence is evidenced through his addition of the numbers in each arm and his evaluation of whether these reached the same total.

Joe’s awareness of the V as two separate lines of 3 is evidenced through both his consistent addition of all three numbers in each arm and through the variability in which this was achieved. Joe either started at the end of an arm and added each number in turn or spotted a known number bond for two cards and then added the third. Frequently each arm was added in a different order; there was no evidence that he viewed the base number as connected to both arms.
Joe’s decisions appeared to be based on the specific numbers in each arm and his facility with the number bonds in question.

Over the course of activity, however, Joe’s approach evidenced a development to structural awareness (Mulligan and Mitchelmore, 2009) in which Joe saw the V as being constructed of two pairs of numbers connected by an additional shared number (Figure 4-2). This reflects a successful disembedding (Owens, 2004) of the different components of the V.

![Figure 0-2 The V as two arm pairs plus a shared base number](image)

Joe’s progression towards this awareness began as he attempted to convince his partner Ash that an attempt at a V was not correct:

<table>
<thead>
<tr>
<th>2:33 Joe: ‘no, yeah 5’ he again indicates the left arm pair (3+2) and then points at the right pair (5+1) ‘and that’s 6 yeah?’</th>
</tr>
</thead>
</table>

Extract 4-1

Joe’s focus on the two arm pairs in Extract 4-1 above reflects a developing awareness that *arm pair totals are the important equivalence* and evidences more productive stressing and ignoring (Gattegno, 2010). This was the first time that this occurred; his argument weakened at this point by an error made when he attempted to add the base card, 4, to each of these totals making a mistake with the addition of 4 to 6 by arriving at the total of 11. He corrected himself but by this time both boys were laughing, and the immediate impact was lost. However, from this point on, Joe used the summing of arm pairs only as a means of checking each subsequent trial, confirming awareness that *the base number does not need to be considered when summing the arms of the V.*
This structural awareness directly led to an awareness of how to engage more productively in the task. In this case it enabled Joe to develop a more efficient way to search for solutions. Joe moved from his earlier random specialising approach (Mason et al., 2010) to the deliberate search for pairs of cards that would have equivalent totals. Joe laid these out (Figure 4-3) in position to form the two arms and only then added the last card at the base to complete the V. This development occurred in the second phase of the activity when pupils were using number cards 2 to 6. The development of his more focused and deliberate trials reflects refinement of his approach in the light of experience (Mason and Johnston-Wilder, 2006; Gattegno, 1987) and enabled solutions to be swiftly found in this phase of activity. By mutual consent, Joe took on the majority of card manipulation at this point in the task; his confidence in so doing evident in his concentration and enthusiasm. When Ash did engage in making trials, he persisted with the use of random specialising.

Figure 0-3 Joe finds pairs with same arm total first
A summary of the development of Joe’s awareness is shown in Figure 4-4.

It is notable that whilst Joe was not utilising any resource that provided enactive representations (Bruner, 1966) of the numbers involved, his use of the number cards demonstrates an enactive engagement with the V shape. Here I refer to Joe’s alteration of the V to reflect his awareness of the V as two pairs plus one central card. The opportunity to manipulate cards supported the development of the structural awareness (Mason and Johnston-Wilder, 2006) evident in Figure 4-3 above. A similar enactive engagement with the V shape and structure is evident in the case of Mia; the development of her structural awareness is presented below.

**Figure 0-4 Joe’s awareness pathway RL1**
Lower attainer Mia, working with higher attainer Ruby, began the task with clear awareness of a more systematic mathematical process:

1:3 Mia: ‘Let’s try first with 1 at the bottom’

Mia’s suggestion above permits the inference of two related awarenesses, both significant in themselves. The first is an awareness of the applicability of the mathematical process of fixing a variable (Posamentier and Krulik, 2009). The second is awareness of base number as useful variable to fix, as opposed to, say, suggesting that they begin with 1 at the top left of the V.

The pair proceeded to fix the base number and manipulate within this constraint until either a solution was found, or they deemed that it was not possible to do so. In the early stages however, the checking of solutions involved summing all three numbers in each arm and thus reflects the partial structural awareness (Mulligan and Mitchelmore, 2009) described above.

Mia’s awareness pathway diagram is shown right (Figure 4-5). As had been the case for Joe, the development of structural awareness (Mulligan and
Mitchelmore, 2009) was first evident in lower attainer Mia as she reasoned that a successful solution had been found:

3:11 Mia sums the right arm pair total saying, with emphasis, ‘6’ then completes addition of 3 by counting on ‘7,8,9’. She then waves her hand (Figure 4-6) above the 5 and 1 of the left arm pair again saying with emphasis ‘6’ then adding on the remaining 3 by tapping the table near the 3, three times as she counted: ‘7,8,9’. She turns and looks to Ruby as she completes this

Extract 4-3

In Extract 4-3, Mia’s vocal emphasis on the 6, the total of both arm pairs, reflects her stressing (Gattegno, 1987) of these pairs as significant and indicates awareness that arm pair totals are the important equivalence. This is further emphasised by her hand gesture indicating the pair of cards totalling six on each side of the V (Figure 4-6). Her repeated, rhythmic counting on of ‘7, 8, 9’ serves to underline the connected awareness that addition of the base number adds the same to each arm. Together her action and utterance evidence a shift in her attention (Mason, 2008a) to dis-embed (Owens, 2004) or separate the arm pair from the base number and to identify an equivalence relationship (Mason, 2011) in the two arm pairs.
Shortly afterwards, Mia used the fingers on each hand simultaneously to identify the arm pairs (Figure 4-7). This enables inference of her awareness that the base number does not need to be considered when summing the arms of the V since Mia did not continue, as was previously the case, to sum the base number to this, instead voicing only the total of the arm pairs, saying simply:

| 4:31 Mia: ‘5 and 5’. |

She also later used the same justification to dismiss a trial as incorrect:

| 4:9 Mia indicates first one arm pair then the other: ‘them two are 7 and that’s 5’. |

As with Joe, Mia’s demonstrated awareness led to more productive ways to engage in the task. Mia’s repeated use of her structural awareness to check solutions enabled efficient and swift solution finding and led the pair to identify numbers which were possible and those that were impossible to have in the base number position.

It is notable, in the light of earlier research findings (Mulligan, 2011; Owens, 2004), that in both the above examples, the lower attainer demonstrated this structural awareness in advance of their partner. At this point both higher attaining partners were not demonstrating a consistent approach to addition. Their continued approach to finding arm totals was to sum a pair of numbers in an arm then add the third, but with no consistent pattern as to which pair of numbers within the three making up
the arm were summed first. This reflects a persisting partial structural awareness (Mulligan and Mitchelmore, 2009) on the part of the two higher attainers at this time.

![Diagram of awareness pathways for Joe and Mia, together with a generalised pathway]

*Figure 0-8 Awareness pathways for Joe and Mia, together with a generalised pathway*
Awareness pathways for Joe and Mia are represented in Figure 4-8 above. Joe’s pathway is shown in the left-hand column with Mia’s shown in the centre column. These reveal remarkably similar trajectories in the development and use of their awarenesses. In each case the progression from a partial structural, to a structural awareness is supported by several contributory awarenesses that arise from the activity in which the pupils were engaged. In these and in other similar awareness pathway diagrams, spaces between boxes in the same column indicates some intervening activity. However, the diagram shows an ordinal progression only; the size of the space and the length of the diagram is not related to the length of time between awarenesses. The right-hand column shows a generalised pathway in which the specific awarenesses have been replaced by more general terms to permit comparisons with subsequent research lessons and with the preliminary lesson.

4.1.2 Research lesson 2

The task ‘Remainders after division’ required pupils to find pairs of number cards from a 1-20 set (see 3.6.5) where division of the larger number by the smaller would leave no remainder. As noted above, data is available for two lower attainers and their respective partners for this lesson. For both lower attainer Mia and her partner Ruby, and lower attainer Rose and her partner Hannah, selection of appropriate card pairs reflected awareness that to divide with no remainder numbers must share a factor/multiple relationship. This meant that the making of successful pairs arose through deliberate selection of cards building on awareness of the required multiplicative relationship (structure) between numbers, rather than through repeated random card pairing and noting the property of successful pairings. Thus, this has been classified as structural awareness rather than awareness of mathematical pattern and property.

Lower attainer Mia and her partner Ruby initially attempted a systematic approach of considering divisors in order beginning with number card 1; this was stimulated by Ruby, but not sustained beyond the consideration of 4 as a divisor and for the remainder of the lesson both pairs employed a random specialising (Mason et al., 2010) approach to the task. In selecting pairs of cards, both lower attainers
demonstrated awareness of doubling/halving relationships, extending into awareness of a range of factor/multiple relationships, as the task progressed.

Development from this initial structural awareness differed from that shown in research lesson 1; arguably this reflects a difference in the nature of the task, its goals and the move to extension activities. Nevertheless, as the task progressed, both lower attainers developed an applied structural awareness. Mia also developed awareness of generality.

In their respective lessons, both lower attainers developed a related structural awareness that a number can act as either a dividend or as a divisor; in both cases this enabled greater flexibility in the search for pairs. The development of this awareness is considered for lower attainer Rose first; her awareness pathway is shown alongside that for Mia at the end of this section.

Prior to the emergence of this awareness, lower attainer Rose, working with Hannah, had been operating a process in which she selected a card, and then paired it by looking for its double. In Extract 4-6 below she has selected 12:

<table>
<thead>
<tr>
<th>Extract 4-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:27 Rose: ‘what’s double 12?’ she looks up and mouths 12, then counts silently, slight, rhythmic nodding of her head. As she does so, Hannah, 12 card in her hand, scans the made pairs touching some and tidying others. Rose finishes her count: ‘24 but we don’t have 24’</td>
</tr>
<tr>
<td>2:29 Rose: ‘we can’t do it with 12. We can only do it with half of 12 because double 12 is 24’ As she speaks she slides the 3 away from the 6 leaving a space, and looks towards Hannah.</td>
</tr>
<tr>
<td>2:30 Hannah doesn’t reply and after a moment Rose takes the 12 from Hannah’s hand and places it next to the six. Immediately Hannah speaks: ‘oh what if we did 3 and 9</td>
</tr>
</tbody>
</table>

Rose’s utterances evidence awareness that the double of some numbers exceeds the card range available. This reflects a shift of attention (Mason, 2008a) from considering 12 as the start number in her process of ‘select number and double it’ (turn 2:27) to contemplating it as the start number of the inverse process (2:29). She was able to apply this structural understanding in manipulating pairs already constructed and in constructing new pairs. Her accompanying movement (turn 2:29) of the 3 card to leave a space next to the 6 in which the 12 could be placed, reflects
confidence in her reasoning and in her capacity to contribute to the task, suggesting a counter to the findings of others (Black and Varley, 2008; Cobb et al., 2009; Bishop, 2012). Her confidence is further confirmed in 2:30 as she reaches over and takes the 12 card from Hannah’s hand. The contribution to exploration that this makes is also illustrated in the same turn as Hannah sees an opportunity incorporating the 3 card that Rose had just liberated from its previous pairing. Rose’s reasoning and action thus contributed to the ongoing exploration in which the pair were engaged. The use, by a lower attainer, of a combination of suggestions accompanied by physical action to effect influence on the task was not an isolated instance and this is further discussed in chapter 5.

For lower attainer Mia working with higher attainer Ruby this same awareness arose as she searched for a partner for the 9 card she was holding:

<table>
<thead>
<tr>
<th>2:14</th>
<th>Mia picks up the 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:15</td>
<td>Mia: ‘Oh ***’, She pauses and stares ahead for a moment.</td>
</tr>
<tr>
<td>2:16</td>
<td>Ruby: ‘6 and 12’ she picks up these cards, forms a pair and stacks them with the other made pairs.</td>
</tr>
<tr>
<td>2:17</td>
<td>Ruby: ‘9 and 18’</td>
</tr>
<tr>
<td>2:18</td>
<td>Mia: ‘What we gonna do about the odd numbers?’</td>
</tr>
<tr>
<td>2:19</td>
<td>Ruby picks up the 18 and repeats: ‘9 and 18 go together’ she turns the 18 to face Mia who is still holding the 9</td>
</tr>
<tr>
<td>2:20</td>
<td>Mia takes the 18 and taps the 9 against it, looking pensive. She pauses, looks at Ruby then says ‘why was I?’ she pauses again and frowns. ‘I was trying to think of half of 9, not double it’ She sighs and shakes her head, pairs the cards and puts them on the table</td>
</tr>
</tbody>
</table>

Extract 4-7 shows the girls working on a strategy of searching for pairs on the basis of doubling and halving relationships. It came after their first attempt, using a range of multiple/factor relationships, had resulted in four cards (8, 13, 17, 19) remaining unpaired. In the extract above, Mia’s awareness that a number can act as either a dividend or as a divisor emerges in turn 2:20 after she becomes aware that halving an odd number will not give a whole number answer (turn 2:18), though this latter awareness is perhaps first realised through her expletive represented by asterisks in turn 2:15. She completes articulation of her own earlier thinking in turn 2:20, the sigh
and shake of her head reflecting that she considered herself foolish. At this stage, Mia’s awarenesses enabled her to maintain her stake in the task and participate in it.

The differing level of confidence and fluency with the identification of multiplication and division facts between the higher and lower attainers in each pair of pupils meant that in the first part of the task, lower attainers’ suggestions and ideas enabled them to maintain a stake in the task and supported development of their own understanding of and fluency with division. However, their higher attaining partners were also demonstrating these same awarenesses and at this stage in the task, higher attainers in both pairs maintained a lead of the task. In both pairs the higher attainer established the impossibility of pairing all cards that concluded this aspect of the exploration. Arguably for this part of the task, the lower attainers gained more from the partnership through exposure to the ideas of their partners (Whitburn, 2001; Leonard, 2001); gains for the higher attainer restricted to those arising from the opportunity to explain and to scaffold thinking for their partners (Roberts, 2016).

Later in their respective lessons for both lower attainers, a further structural awareness was developed and in each case this extended beyond that demonstrated by the higher attainers in the pairs. Rose and her partner had been tasked with finding pairs that gave a remainder of 1. Higher attainer Hannah had articulated an approach, which both girls employed, of selecting a number, doubling or tripling it and adding 1. Rose appeared confident to apply this strategy. Work proceeded to the

Figure 0.9 Rose selects the 15

approach, which both girls employed, of selecting a number, doubling or tripling it and adding 1. Rose appeared confident to apply this strategy. Work proceeded to the
point shown in Figure 4-9 where five pairs have been made from the twenty cards, leaving ten cards on the table. Rose has just replaced the 10 card. She looked round the assembled cards before picking up the 15. She asks:

Extract 4-8

3:22 Rose: ‘Do you know a times table that makes 14?’

In Extract 4-8 Rose demonstrates awareness that consideration can begin with the dividend rather than the divisor. Importantly however she also demonstrates awareness of the inverse operation. She has, in effect reasoned about the relationship between the dividend and the divisor in a division with remainder 1 relationship and, rather than beginning with a number, multiplying by a selected factor and adding 1, she has begun with 15, subtracted one to reach 14 and is looking for factors of 14. This reflects a shift in the way in which Rose is attending to the mathematical relationship (Mason, 2011; 2003) between the numbers concerned. Shortly afterwards, Rose independently answers her own question by selecting 7 and 15 as a pair. This awareness marks a change in that from this point onwards Rose appeared to work with greater confidence and operated as a more equal partner in the task, contributing consistently to subsequent realisations about how to successfully pair the majority of the number cards.

Whilst engaged in the same extension task of finding pairs with a remainder of 1, rather than pairing cards up, Mia and Ruby identified a successful pair, recorded it and then returned the card pair to the set of cards laid out. This afforded an opportunity for Mia to show a different style of considering possibilities which led to important reasoning about remainders of one when odd numbers are divided by 2:

Extract 4-9

3:17 Mia holds the 2 next to the 11: ‘look 2 and 11’. Ruby replies ‘yep’ and Mia then briefly touches the 12 card before moving the 2 next to the 13; ‘2 and 13’.

3:18 Mia watches and waits for Ruby to record then moves to hold the 2 next to the 15 touching her right hand on to the 14 as she moves the 2; ‘2 and 15’.

3:19 She waits again then repeats the move, again touching her hand to the 16; ‘2 and 17’ She waits again and says: ‘cos it goes into all of the..’

3:20 Ruby interjects: ‘yeah’

3:21 Mia adds the final pair: ‘and 2 and 19. Cos it goes into all of the even numbers’
Mia speaks with increasing urgency in Extract 4-9 as she progresses through turns 3:17 to 3:21. She waits each time for Ruby to record the division calculation, underlining the importance she attributes to her realisation. Mia’s awareness in this extract begins with an awareness of pattern, a predictable regularity in the division of each of the odd numbers (turns 3:17 to 3:20) by two (Mulligan and Mitchelmore, 2009). This progresses into an articulation of awareness of the underlying generality as she moves from considering the individual instances to considering the relationship that exists (Mason, 2003) in its general form. The awareness that any odd number divided by 2 leaves a remainder of 1 is articulated as she reasons: ‘cos it goes into all the even numbers’ (turn 3:21). This stands in some contrast to a claim made a short while earlier by Ruby that any odd number divided by any even number would result in a remainder of 1. This latter claim had been based on a small set of recorded calculations but was not tested further to challenge or confirm its validity (Mercier, 2011; Lithner, 2000a).

As was the case with research lesson 1, the resources of this task were not enactive representations of the numbers involved but the presence of number cards did afford the opportunity to engage physically with the ideas involved. Manipulation of cards enabled identification of numbers that had and had not been used and supported flexibility in the pairings made. In addition, Mia’s engagement with the cards in Extract 4-9 above supported the development of the awareness of generality expressed. Her movement of the 2 card sequentially to rest next to each odd number (turns 3:17, 3:18 and 3:19 above) was accompanied by a brief touch of each of the even numbers that interspersed them. I would argue that represented enactive engagement with the mathematical idea underpinning the awareness expressed and supported her confident articulation in 3:21 above: ‘cos it goes into all of the even numbers’.

Mia articulated a second generality in this same critical sequence saying, quietly: ‘you can’t do 1 with 3, no.. with anything’. This statement reflects awareness that division by 1 does not leave a remainder and this single statement shows the development from awareness of a single case to awareness of a generality (Mason, 2008a) Both of
Mia’s generalities represent awareness about division that was not evident at the start of the activity.

Awareness pathways for Rose (left) and Mia (centre) are shown below (Figure 4-10). In this instance whilst their pathways towards what I have described as an applied structural awareness are similar, Mia’s later development of awareness of generality enabled her to exert a greater impact on task progress and arguably led to deeper understanding about division and remainders through articulation of these important generalities. I interpret the development of structural awareness differently to that identified in research lesson 1. In the former, the development of structural awareness was closely linked to Mulligan and Mitchelmore’s (2009) categorisation of partial structural awareness and structural awareness. In research lesson 2, I would argue that awareness of structure was not developed into a different category of awareness, but that it was deepened through application in addressing complexities of selection and thus became an awareness more flexibly applied. The generalised pathway is slightly harder to construct in this instance since the later development for each lower attainer differed. I have represented the aspects demonstrated only by Mia in grey to indicate that these arose out of the particular way in which Mia and her partner engaged in the second part of the task; this pathway is shown in the right-hand column as before.
Structural awareness: factor/multiple relationship

Awareness that the double of some numbers exceeds the range available

Awareness that a number can act as a dividend or divisor

Awareness that consideration of number pair can begin with the dividend rather than the divisor

Awareness that the double of some numbers exceeds the range available

Awareness that a number can act as a dividend or divisor

Awareness that consideration of number pair can begin with the dividend rather than the divisor

Applied structural awareness

Awareness of generality

Contributory awarenesses

Awareness pathway for Rose and Mia together with generalised pathway

Figure 0-10 Awareness pathway for Rose and Mia together with generalised pathway
4.1.3 Research lesson 3

Three of the four lower attainers developed and demonstrated a significant task related mathematical structural awareness during the course of their activity. In each case this preceded their higher attaining partners demonstrating this same awareness. The fourth lower attainer, Daisy, did not convincingly demonstrate this awareness and this is discussed in section 4.3. For each of the three lower attainers demonstrating this structural awareness this reflected a shift in the way that the numbers in the bags were attended to (Mason, 2003). The shift required was to move from attending to each number holistically as a single quantity (e.g. 4, 7) to seeing it as a multiple of 3 plus 1. This is categorised as a structural awareness because it reflects the underpinning organisation (Mulligan and Mitchelmore, 2009) of the numbers involved; the consistent relationship present in the set. Awareness that all numbers in the set could be conceptualised in this way enabled pupils to explain why adding three such numbers together resulted in a total that was a multiple of 3. Unlike the previous research lessons, this was a shift to a structural awareness rather than the development from one level or type of structural awareness to another. An awareness pathway representing all three pupils’ progression is shown at the end of this section.

For all three lower attainers structural awareness developed after the mathematical property awareness that all totals are multiples of 3 had been established (see 4.2.2). In each pair the resource Numicon was used to represent this commonality with pupils constructing models in which three of the numbers from the set were selected, combined together and then either covered with yellow Numicon 3 pieces (Figure 4-11, left hand image), or with yellow Numicon 3 pieces lined up next to it (right hand
This activity enabled lower attainers to demonstrate awareness of how the multiple of 3 property of the totals can be represented enactively. In all classes, teacher emphasis on explaining this outcome ensured prolonged attention to this aspect of the task.

Rose’s structural awareness that each of the numbers in the bag can be seen as a multiple of 3 plus 1 was supported by her use of Numicon. Figure 4-12 shows Rose’s hand partitioning the pink Numicon 7 piece into six and one. A few moments later, this reasoning developed between the two girls to confirm that 4, 7 and 10 are all one more than a multiple of 3. However, when it came to extending this thinking to the 1 piece Numicon, the higher attainer Hannah was stuck:

3:6 Hannah: ‘yeah cos four is, one less is three and seven, one less is six’. Picking up a Numicon 1 piece and holding it up, she says in impassioned tones: ‘And the ‘one’ I have no idea about’

3:7 Rose is sitting sideways on her chair with one foot on the chair, hands on her knee, chin resting on her hands. She is looking at Hannah. She lifts her head and takes the ‘one’ piece from Hannah and is about to speak as Hannah exclaims: ‘the one is just, a one!’

3:8 Rose jumps in quickly as Hannah finishes: ‘no, no, no, if it was one less it would be zero and there would be nothing there. And that would be a multiple of 3. Zero, three, six…’ she waves her hand right and left as she begins a rhythmic times table count.

3:9 Hannah, excitedly and with mouth open wide: ‘oh yeah!.. Oh yeah!’ Rose is smiling very broadly now

Extract 4-10

In Extract 4-10 exposure to the ideas of her higher attaining partner (Leonard, 2001; Whitburn, 2001), specifically Hannah’s lead in considering other numbers in the set
(turn 3:6), arguably supported Rose in developing the awareness that she demonstrated in turn 3:8. In turn 3:8 Rose shows that she can follow and extend the pattern of considering one less than the number in the set. Moreover, she can use the same form of articulation as her partner in justifying her thinking. I consider this to be evidence of Rose operating at the same level of mathematical thinking as her partner at this time. Her awareness that \textit{1 has an equivalent relationship to multiples of 3 as do the other numbers in the set (i.e. that all are one more than a multiple of 3)} is one of the most powerful and significant of the study. Its significance lies in its impact on the task for Rose and her partner and because no other pupil in a focus pair, higher or lower attainer, demonstrated this particular awareness. Specifically, the identification of shared commonality across all four of the numbers supported the awareness of generality that ensued (Towers and Martin, 2014) and their confidence in it. Significantly also, the response from both girls to this awareness; excitement from Hannah and delight from Rose, served to give the task further momentum and raised Rose’s status in the task.

This identified commonality led directly to a construction using Numicon being made (Figure 4-13). This construction reflects awareness of how the structural organisation of the numbers in the set as multiples of 3 remainder 1 can be represented enactively.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure_4-13.png}
\caption{Three fours each represented as a three remainder 1}
\end{figure}

In this image three Numicon 4 pieces have been set out. Each 4 piece is covered with a 3 piece and a 1 piece. The 1 pieces have been covered with coloured erasers with small pieces of paper labelled R placed over them. This construction both reflected
the girls’ understanding of these as remainders and supported their later reasoning about the result of adding three such remainders together (Fyfe et al., 2014; Marley and Carbonneau, 2014). Later, both demonstrated the structural awareness, using addition of three fours as an example that when three numbers that are all one more than a multiple of three are added together the ‘remainder’ from each one makes an additional 3 thus making the total a multiple of 3.

In the parallel lesson, lower attainers Mia and Joe working with their respective partners separately demonstrated awareness that three of the numbers in the bag are one more than a multiple of 3. The three numbers for which this property was identified were 4, 7 and 10. Neither pupil extended their thinking to the fourth number of the set, 1. The development of these related awarenesses is similar for both pupils; the case of Mia is reported here:

<table>
<thead>
<tr>
<th>Extract 4-11</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:1 Mia takes a sharp in breath, smiles and says ‘ah’. She points at the Numicon: ‘in all three numbers, oh, um, they have like, one left over’</td>
</tr>
<tr>
<td>2:8 Mia picks up the Numicon 4, putting it on the table in front of her and places a 3 on top of it. ‘If you like take 3 away from that you have one left’</td>
</tr>
<tr>
<td>2:10 Mia: ‘you get the same with this one’ she puts two threes on the 7; Ruby leans back, smiles and releases a long sigh. Mia continues: ‘and you would get the same with 10 as well’ she puts three 3s on the 10 showing the one left over</td>
</tr>
<tr>
<td>2:11 Mia (on the left of Figure 4-14 below) orients the Numicon so that the remainders are close together and places a further Numicon three in the space created; ‘it makes another three and it ends up being in the three times table’</td>
</tr>
</tbody>
</table>

*Figure 0-14 Mia adds another 3 piece to the three remainders*
In Extract 4-11 Mia identifies the relationship between each of the numbers and multiples of 3 in turns 2:8 and 2:10. Although she considers each number separately, her articulation in 2:1 references ‘all three numbers’ and thus suggests that the commonality between them that paves the way for the generality has already been identified (Towers and Martin, 2014). Her modelling with the Numicon in 2:8 and 2:10 is thus part of communicating to her partner what she has already realised herself (Houssart, 2004). It represents her awareness of how the structural organisation of the numbers in the set as multiples of 3 remainder 1 can be represented enactively. Ruby’s smile and sigh (2:10) indicates that she has come to the same awareness. Mia’s modelling leaves the ‘one left over’ uncovered by Numicon and this supports her later reasoning (Fyfe et al., 2014; Marley and Carbonneau, 2014). She subsequently oriented the 4,7 and 10 pieces as shown on the left of Figure 4-14 above (and reconstructed for clarity in Figure 4-15) so that the uncovered ‘remainders’ are adjacent to each other.

Her positioning of the additional 3 piece (shown in her hands in Figure 4-14 above) in the space created provides a strong representation of her awareness that when three numbers that are all one more than a multiple of three are added together the ‘remainder’ from each one makes an additional 3. Her articulation in turn 2:11 is confirmation of her awareness of this relationship.

For all three lower attainers demonstrating this awareness, availability of Numicon as a resource enabled enactive engagement with the multiple of 3 plus one structure
of individual numbers and with the outcome of summing three such numbers. The value of representations in supporting awareness of structure and construction of reasoning has been established in earlier research (Marley and Carbonneau, 2014; Fyfe et al., 2014; Mason and Johnston-Wilder, 2006; Houssart, 2004); the particular value of Numicon is evidenced by its successful use in this lesson by three of the four focus lower attainers. The fourth lower attainer did not have access to the same resource in her parallel lesson.

Figure 4-16 overleaf shows the awareness pathway for all three pupils on the left hand side. The grey shaded cell represents the awareness only demonstrated by Rose; all other awarenesses were common to all pupils. The generalised pathway is shown on the right. As noted above, this pathway begins with a mathematical property awareness, and the inclusion of awareness of how the relationships could be represented enactively is included here because of their significance in developing the reasoning that ensued.
A further awareness arising in this lesson is worthy of consideration, since for Joe and Ash this led their exploration down a very particular path. The awarenesses involved are not included in the pathway above, being specific to this pair and arising from independent conjecturing from the higher attainer Ash. The inclusion is justified nevertheless because it demonstrates the contribution of the lower attainer in the development of reasoning. At this point in the task the pair have demonstrated
awareness that all totals are multiples of 3 and are beginning to consider why this is the case:

<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:19</td>
<td>Ash, quietly: Oh, OK, yep’ as if he has realised something. He turns towards Joe who has returned gaze to the board. Ash pauses and Joe turns again to look at him as Ash raises his pen.</td>
</tr>
<tr>
<td>1:20</td>
<td>Ash: ‘cos you can only pick three numbers can’t you?’</td>
</tr>
<tr>
<td>1:21</td>
<td>Joe nods</td>
</tr>
<tr>
<td>1:22</td>
<td>Ash continues: ‘and they’re in the three times tables’</td>
</tr>
</tbody>
</table>

In Extract 4-12 Ash reasons that the totals reached are all multiples of three because three separate numbers have been summed on each occasion. This conjecture, whilst incorrect, was common across all classes and represented a connection noted between a property of their outcomes and the problem scenario. As Ash began to record this conjecture on the shared recording sheet, Joe looks hard at the board before leaning back, stretching, and then leaning forwards to speak:

<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:4</td>
<td>Joe leans towards Ash, who is still writing, head down, and speaks, looking at him: ‘Ah but Ash if you were only allowed to choose 2 it would be in the one times table only’ Ash stops writing and sits up</td>
</tr>
</tbody>
</table>

Joe has demonstrated (Extract 4-13) awareness that the conjecture does not hold true if only two numbers are summed. The total is not in the two times table as would be expected. Ash attends to what Joe says, shown by the pause to his recording. He looks towards the board with a puzzled expression and selects two numbers. Joe takes these two numbers on as an example to demonstrate the validity of his argument:

<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:9</td>
<td>Ash: ‘4+7’</td>
</tr>
<tr>
<td>2:10</td>
<td>Joe: ‘4+7=11’</td>
</tr>
<tr>
<td>2:11</td>
<td>Ash exhales, looks pensive and turns towards the board. Joe maintains his focus on Ash, frowning: ‘and what calculation goes into 11, only 1, and 11’ Joe looks towards the board.</td>
</tr>
<tr>
<td>2:12</td>
<td>Ash frowning, continues to gaze ahead. He mutters: ‘so 12....22’,</td>
</tr>
<tr>
<td>2:13</td>
<td>Joe looks at Ash and repeats: ‘4 and 7 is 11’</td>
</tr>
<tr>
<td>2:14</td>
<td>Ash asks: ‘is 22 in the four times table?’ He drops his pen and puts his hand to his face</td>
</tr>
<tr>
<td>2:15</td>
<td>Joe counts on his fingers: ‘4,8,12’ and is cut off by Ash saying: ‘no its not’</td>
</tr>
</tbody>
</table>
In Extract 4.14 Ash’s sigh in turn 2:11 is an acceptance that Joe’s reasoning is correct. Joe draws on his knowledge of 11 as a prime number (turn 2:11) to reason by the individual case (Mueller and Maher, 2009) that since its only factors are 1 and 11, it cannot be in the two times table. Whilst the set of numbers from which they are selecting does include those which could result in a total in the two times table (e.g. 10+4; 7+1) Joe is correct that this will not always be so. Thus Ash’s reasoning that the earlier totals are multiples of three because they are adding three of them does not hold true if two numbers are summed. As with the example of Rose above, I contend that this extract reflects Joe operating at an equivalent level of mathematical thinking as his higher attaining partner at this time. His capacity to follow Ash’s line of thinking, to artfully specialise (Mason et al., 2010) in selecting a related example, to use this to refute the conjecture and to justify this all serve to support this contention. This exchange and Joe’s challenge to Ash’s conjecture impact significantly on the work of this pair; indeed this theme productively dominates their thinking for a large part of the remaining activity time. It is the topic that they choose to feedback during the mid-lesson plenary, in which the class teacher explicitly notes the merit of Joe’s reasoning.

4.1.4 The preliminary lesson

Data from the preliminary lesson demonstrated that lower attaining pupils became aware of and were willing to assert constraints arising from the task structure. These have been classified as task related structural awarenesses as they arose through consideration of, or with reference to, the resources in hand rather than arising from consideration of outcomes already obtained. In two of the four pairs, clear assertion was made to this effect by the lower attainer (both Mia and Joe); in a third (Rose) it was implied through gesture and some speech, but its security cannot be confirmed. In the fourth (Daisy) awareness demonstrated by the higher attaining partner was demonstrably shared through subsequent action.

Both Joe and Mia directly articulated awareness of a significant structural constraint. This referred to the upper limit possible on the carriage dice tops arising from the fact that the largest number on the funnel die is 6. Since the carriage tops must sum
to the number shown on the funnel die, the carriage tops cannot sum to more than six.

The following exchange took place between Joe and his higher attaining partner Ash within a few moments of the paired activity beginning:

**Extract 4-15**

<table>
<thead>
<tr>
<th>Joe holds two dice in his hand and is beginning a trial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:3 Joe: ‘Well you can’t make 7 so,’</td>
</tr>
<tr>
<td>1:4: Ash: ‘yeah so’</td>
</tr>
<tr>
<td>1:5 Joe: ‘So let’s try 3 and 2’. He places the two dice on the table, 2 and 3 uppermost</td>
</tr>
</tbody>
</table>

It is evident that Joe’s comment in Extract 4-15, turn 1:3 guided his immediately subsequent actions; in turn 1:5 Joe’s speech shows a logical connective ‘so let’s try 2 and 3’ together with the use of random specialising (Mason et al., 2010). However, from his utterance in turn 1:3, it is not clear whether Joe’s comment reflects awareness about the total 7 alone, or whether the comment is underpinned by a more general understanding (Mason, 2008a). The dialogue that followed later (extract 3 below), evidences that the latter is the case:

**Extract 4-16**

<table>
<thead>
<tr>
<th>Ash has two dice arranged with faces 5 and 5 uppermost and a third with 3 uppermost onto which he is positioning the funnel dice (FD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:14: Joe looks over at the top faces of the dice: ‘How are we supposed to make 10, there’s no 10 on the dice’. He turns to look at Ash</td>
</tr>
<tr>
<td>2:15: Ash pauses, still holding FD which has just been positioned on the train with 4 on top: ‘Oh yeah, we can’t go above 7 can we’. He looks ahead, hands still on the train</td>
</tr>
<tr>
<td>2:16: Joe pauses for a second then looks at Ash again: ‘we can’t go above 6’</td>
</tr>
</tbody>
</table>

Here Joe’s comment in Extract 4-16, turn 2:14, presents a further specific example of an impossible total. Unlike in the earlier expression (Extract 4-15, turn 1:3) here he gives a reason for its impossibility, referring to there being no 10 on the dice. This second, more forceful assertion is sufficient to effect a pause in Ash’s action and he draws on Joe’s reasoning to form a statement of reasoning of his own in turn 2:15, an example of the integration of Joe’s assertion into his own thinking (Mueller et al., 2012). Joe responds with a modification to Ash’s reasoning and a correct statement in turn 2:16. This last statement evidences direct reasoning (Mueller and Maher,
2009) on Joe’s part and awareness that since the dice represented the values 1-6, no solutions where carriage totals summed to greater than six are possible in creating viable trains according to the rules of the activity.

As had been the case in research lesson 1, Mia began the task with a demonstration of a mathematical process awareness:

<table>
<thead>
<tr>
<th>Extract 4-17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mia and Ruby are working with four dice, three pink and one orange</td>
</tr>
<tr>
<td>1:2. Mia: ‘Shall we start with ones?’ Mia orients all three pink dice in line with rule 1, so that ones are on top.</td>
</tr>
</tbody>
</table>

This suggestion of fixing a variable (Posamentier and Krulik, 2009), a form of systematic specialising (Mason et al., 2010) was not built on (see chapter 5) and work proceeded using a trial and improvement approach. A little later, whilst observing Ruby’s activity with the dice, Mia noted the same structural constraint as Joe, albeit at a later point in pair activity:

<table>
<thead>
<tr>
<th>Extract 4-18</th>
</tr>
</thead>
<tbody>
<tr>
<td>4:1 Mia, looking towards the train: ‘it can’t be more than six though because these dices (touching the FD) don’t go more than six</td>
</tr>
</tbody>
</table>

In the remaining pairs Rose appeared to show awareness of this same constraint when protesting to her partner through gesture that a construction was impossible.

<table>
<thead>
<tr>
<th>Extract 4-19</th>
</tr>
</thead>
<tbody>
<tr>
<td>3:12 Poppy picks up a carriage die and begins to turn it to find the matching face (5)</td>
</tr>
<tr>
<td>3:13 Rose points at the tops of the funnel die and the existing single carriage die. They both show 6: ‘that won’t work’</td>
</tr>
</tbody>
</table>

In Extract 4-19 Rose demonstrates awareness that this particular construction cannot be correct. However, it is not clear whether her objection is because it breaches the rules of the activity that the two carriage dice must sum to the funnel die, or through awareness of the structural constraint meaning that it is impossible to have a six on one of the two carriage die. Subsequent articulation of the same objection by Rose did not reveal the basis of her objection further. The final focus lower attainer, Daisy, did not articulate this awareness but worked in line with it following its articulation by her higher attaining partner.
A further structural constraint arises from the positioning of numbers on opposite faces of the dice. Two of the four focus lower attainers demonstrated clear awareness of this constraint. As Ruby struggled to improve an arrangement Mia noted that:

4:29 Mia: ‘yeah, cos 1 is only opposite, 1 is only er opposite 6’  
Extract 4-20

In Extract 4-20 above Mia provides commentary on the challenge in which Ruby is engaged as she tries to manipulate a die unsuccessfully. The likely roots of this awareness were evident early on in the activity when Mia was observed picking up a die and turning it slowly in front of her. She had placed fingers on one face and turned the die over to look at the opposite faces. At the time she had said nothing, and this awareness was not conclusively guiding her actions until she made the comment above. She must however have been aware of it prior to this utterance since she was not either holding or within sight of dice faces at this point in a way that could have revealed this.

Daisy noted the same structural constraint whilst watching her partner trying to rotate a die to improve an arrangement. As her partner worked Daisy said:

Daisy: ‘what’s opposite’, then ‘oh, 1’ (opposite 6)  
Extract 4-21

Further evidence that this constitutes awareness beyond the specific example of 1 and 6 comes from her explanation to her class teacher a few moments later of why the arrangement they are working on cannot be made to work:

Daisy: ’3 and 3 are touching (on adjacent faces) and we need 4 there (indicating the top face) and 4 is there (indicating the side of the die), because 3 is opposite 4’  
Extract 4-22

Common to all the above is that despite their validity, these awarenesses had little impact on task progress. This is discussed in chapter 5. A curtailed awareness pathway results and this is presented below (Figure 4-17) reflecting the example of the constraint that carriage totals cannot exceed 6. Here the left-hand side reflects the pathway for Mia and Joe since for these pupils the nature of their awareness (unlike that of Rose) is clear. The right-hand diagram represents the generalised
pathway. Mia’s mathematical process awareness is included and shaded grey to indicate that this was not common to all focus lower attainers. It is also not clear if or how it contributed to her later structural awareness. The pathway for the second awareness illustrated above relating to numbers opposite each other results in the same curtailed pathway.

Figure 0-17 Awareness pathway Joe and Mia together with generalised pathway PL
4.1.5 Section conclusions

The generalised pathways for the three research lessons together with that of the preliminary lesson are shown overleaf (Figure 4-18).

Together these generalised pathways indicate that the intervention had a strong impact on the development and use of lower attainers’ structural awarenesses. Evident in all three pathways relating to the research lessons is the degree of development that lower attainers made in the awarenesses expressed; the contributory awarenesses arising from activity enabling progression from one awareness to the next. Evident also is that in each research lesson this development had a clear impact on task progress. This stands in stark contrast to the outcomes of the preliminary lesson in which awarenesses were not built upon and task impact was minimal.

I have also shown that the development of structural awareness follows a range of pathways. Examples here show progression from one level of structural awareness to another (research lesson 1); from a structural awareness to the applied use of this and to some awareness and expression of generality (research lesson 2); and from a mathematical property awareness towards a structural awareness (research lesson 3). The strong degree of similarity between lower attainers in different pairs and in different classes that has been shown in the data presented suggests that these different pathways are task related rather than pupil specific; further research is needed to explore this conjecture.
Figure 0-18 Generalised awareness pathways from RL1,2,3 and preliminary lesson
Analysis in this section has shown that lower attaining pupils can demonstrate
awarenesses that constitute valuable mathematical contributions to the progress of
tasks. Furthermore, these contributions frequently arose in advance of their higher
attaining partners demonstrating the same awarenesses. Comparative data from the
preliminary lesson suggests that where structural awareness on the part of the lower
attainer is either not evident or not productive, then factors other than their capacity
to contribute in this way need to be examined. These ideas are further discussed in
chapter 6.

Lastly, data also indicates that for these lower attainers the importance of being able
to make contributions to the task is significant in the status afforded and the
confidence that emerges. This is evident in their responses at the time and frequently
in the responses of their partners also. It is also evident in occasional comments aside
from lower attainers at the end of lessons. One such comment was made to me by
Mia, and with evident pride, at the end of research lesson 3 as I packed equipment
away but before all voice recorders had been switched off:

Mia: I thought Ruby would be the first one to figure out like, find out like it's like there's
one left over like, but I did.

Extract 4-23

For Rose a comment from the same lesson came as the class teacher invited the class
to indicate using a thumbs up/down whether they had understood the reason why
totals were all multiples of three. Rose has heard someone near her say that they
'kind of got it' Rose says:

Rose: I didn't kind of get it, I got it. She puts both thumbs up and smiles broadly.

Extract 4-24
4.2 Awareness of mathematical pattern and property

This section addresses the following finding:

b) The focus on noticing had limited impact on lower attaining pupils’ awareness of the mathematical patterns and properties of their findings and results.

As noted in 4.1.2, the nature of pupils’ pre-existing awareness of multiples and factors meant that awarenesses leading to the making of number pairs in research lesson 2 are categorised as structural awareness. In this lesson, important outcome properties relating to prime numbers could arise (see appendix A), however either the higher attainer first articulated these, as in the case of lower attainer Mia and partner Ruby, or as was the case with lower attainer Rose and partner Hannah, neither pupil was familiar with the concept of primes and the class teacher introduced the concept and its vocabulary once the set of cards that could not be paired had been identified. Awareness of specific multiple/factor relationships also demonstrated in research lesson 2 is not a focus for analysis. Thus, analysis in this section addresses research lessons 1 and 3 only.

Throughout this section awareness pathway diagrams are not included. This reflects the limited occurrence of pattern and property awareness and that little development from expressed awarenesses was evident.

4.2.1 Research lesson 1

The key mathematical property awareness that enabled progress in the task beyond successful searching for solutions is that for the original version of the task (cards 1-5), all successful solutions have odd numbers at the base of the V. Awareness of this outcome permits mathematical reasoning relating to the connection between this outcome and the odd/even properties of the whole card set, and permits prediction as to the property of the base numbers when the card set 1 to 5 is substituted for a 2 to 6 set. The research team anticipated that this base number property awareness would arise either from hands on practical trialling of different arrangements of the
number cards, or that it would emerge as a noticing as pupils recorded their successful solutions.

In the event, neither the higher nor the lower attaining pupils in any of the focus pairs demonstrated this awareness before direct prompts by the class teachers. Following such prompts it was the higher attainer in each pair who first articulated the awareness.

In all classes, once this essential outcome property had been identified, pupils were asked to explain why this was the case. Whilst all pairs engaged in this discussion, the only pair to arrive at a coherent explanation in the time available (before a whole class plenary which addressed the issue) was lower attainer Daisy and her higher attaining partner Jess. In this case Daisy articulated the explanation. This followed a class teacher intervention at their table in which the teacher had urged them to think about:

<table>
<thead>
<tr>
<th>Teacher 2: ‘What kind ..not necessarily the value of the numbers but the kind of numbers I've left you with.’</th>
</tr>
</thead>
</table>

In Extract 4-25 the teacher’s reference to the numbers the pupils were ‘left with’ relates to the four numbers forming the arms when an impossible arrangement, a V with 2 at its base, is used as a basis for explaining why an even number cannot be the base number. Jess led consideration of the properties of each of the remaining numbers (1, 3, 4, 5). Interestingly, Jess’s attention was focused on whether these numbers were all prime or were all square. As this consideration proceeded, Daisy, who had been manipulating and stacking the Numicon pieces representing three of these four numbers, spoke with some urgency:
**Extract 4-26**

3:24 Daisy: ‘Oh my god... no, no no...’ Daisy takes the 4 Numicon from Jess’s hand and includes it in her stack of Numicon pieces. She now has the pieces 1,3,4,5 stacked together; ‘Look add all of these up together, look 5,4,3,1’ Jess looks over at Daisy

3:25 Daisy: ‘add all of these up together’. She has her head down and lifts and firmly replaces her stack of Numicon pieces back down on the table; ‘all of the numbers that we’ve got’

3:26 Jess turns back towards the cards on the table, pulling each one towards her as she adds ‘4..9..10.13’ she turns back towards Daisy as she emphasises 13. Daisy still has her head down

3:27 Daisy lets go of her stack of Numicon and turns slowly towards Jess, lifting and pointing her left index finger as she speaks: ‘13 is odd..... so you can’t split it up, like you can with 12 and 14’

Daisy’s utterance in Extract 4-26, turn 3:24, reflects an important change in how the explanation is being sought and provides an alternative focus in stressing and ignoring (Gattegno, 2010). Up to this point, properties of each individual number (of the set 1, 3, 4, 5) were being considered; Daisy (turn 3:24) now begins to consider a property of their total. Her actions and speech (turn 3:25), conveying a quiet insistence, reflect her confidence in her awareness that 13 is an odd number. In turn 3:27 she reasons about this property awareness and demonstrates awareness that with 2 as the base number *two equivalent arms cannot be made* in the V. Her utterance in turn 3:27, whilst not making explicit reference to the forming of equal amounts is nevertheless a careful articulation of this awareness. This was a significant moment in the progress of the task, constituting the formation of direct reasoning (Mueller et al., 2012) about the numbers involved. This enabled the pair to share this finding in a subsequent class discussion and to apply the same reasoning in the second part of the task.

**4.2.2 Research lesson 3**

In this lesson, the key outcome property awareness that all totals were multiples of 3 was successfully gained by all focus pairs relatively swiftly. However, in three of the four pairs the higher attainer noticed this commonality first; for some pupils, other features of the outcomes were afforded attention before this key awareness
emerged. Again, teacher questioning provoked the key awareness for two pairs, including the pair in which the lower attainer noted the shared property first.

Prior to identifying multiples of 3, Mia and Ruby had noticed other patterns in the totals that they had recorded. These arose specifically because of the numbers they had chosen in their first four selections (Figure 4-19).

Ruby noted a difference of 9 between all of their first four totals; Mia attended to pattern in the digits obtained noting that the units digit diminished as the tens digit increased. At the point at which the class teacher arrived at their table they had recorded all solutions shown in the image above but were only able to report on the patterns noticed for the first four solutions. The class teacher asked them to:

Teacher 1: ‘look at the totals and see what they have in common’

Extract 4-27

This was sufficient to effect a short pause in which both girls looked at the sheet; Mia subsequently noting the commonality after a few seconds and being the only lower attainer to do so first.

Across all lessons, I observed that higher and lower attainers alike rarely independently attended to the properties of the outcomes of their trials. Teacher prompting to look at results was common. Where such independent examination did take place, pupils tended to look at properties of individual outcomes rather than considering commonality across a set of results. An example of recording from Hannah (HA) and Rose (LA), research lesson 3 is below:
In this lesson, Rose and her partner used their recording sheet to identify all properties they could think of for each of their totals (Figure 4-20 above). Their inscriptions on the right-hand side of the sheet note that 15 is in the five times table, that 18 is in the 2, 6 and 18 times table and that 21 is in the 3 and 7 times table. Here, the two girls did not independently make the shift in attention (Mason, 2011) required to consider the outcomes as a set.

4.2.3 The preliminary lesson

Given the strong affordances (see appendix A) in this lesson for awareness of a range of mathematical properties associated with the outcomes of successful trials, the relative paucity of awarenesses expressed is notable. The following represent the totality of awarenesses demonstrated under this category by either the higher or the lower attainers:

\textit{Mia, is checking that a new solution found is a novel one and not a repeat}

5:2: ‘Yeah and we haven’t got it (this solution already) cos we haven’t had a single one with 5 at the top’

Extract 4-28

A few minutes before this Mia had also noted quietly that most of their solutions had 4 at the top (on the funnel die).
Daisy paid more attention to the outcomes than any other of the focus lower attainers or their partners, but this did not occur until quite late in the activity; her first mention comes some twenty minutes into the task:

Daisy glances over at the recording sheet; two solutions both totalling 4 (on the funnel die) have been found:
3:1 Daisy ‘Let’s find something that equals not 4 today’

Later in the lesson Daisy again looked at the results and identified that they had not found one with 3 or 2 at the top and together the girls decided to aim for a three total. This represents the only direct use of outcome awareness to guide trials that is evident across the four pairs. Joe and his partner overtly noted no features of the funnel die top face in their successful solutions, checking only that the appearance of them when all elevations of the train were considered was different. Rose and her partner found no solutions during the lesson.

Whilst it is possible that focus lower attainers and indeed their partners had been aware of features of their outcomes from a much earlier stage, none, if this is the case, had felt that these were awarenesses worthy of comment. Aside from Daisy’s comment above, the absence of discussion relating to the solutions found meant that none of the focus pairs contemplated together how many solutions with each ‘funnel’ or ‘top’ number were possible; there was no sense that any of the pupils had any notion of when this activity might end. Across all focus pairs pupils appeared content to keep on searching for solutions without any concern as to the total number possible.

4.2.4 Section discussion and conclusions

Data presented above has explored the finding that the intervention had limited impact in enabling lower attainers to develop and express awareness of mathematical property and pattern in their results. This is despite the mathematical properties of number that were in question during the research lessons (odd and even numbers and multiples of single digit numbers) being comfortably within the established knowledge base (Ball and Bass, 2003) and experience of all pupils
involved. This was an initially surprising outcome; its contrast with the finding in relation to awareness of structure quite stark.

In seeking to account for the limited impact of the intervention on lower attaining pupils’ awareness of pattern and property, data suggests that teacher intervention intended to provoke awareness of properties of outcomes requires more than a general invitation of the kind: ‘look at your results and see what you notice’. In line with earlier findings (Gresalfi et al., 2012) teacher questioning needed to define the space within which pupils’ attention should be focused. An example of Mia and Ruby’s response to a general invitation is below:

| TC1:1 Teacher: ‘Is there anything that you have noticed that you could record?’ |
| TC1:2 Ruby: ‘that, I don’t,…..we’ve noticed that you, it’s impossible to have one with 2 at the bottom’ |
| TC1:3 Teacher: ‘Ok so what is possible then?’ |
| TC1:4 Mia: ‘Um, 3,5 and 1 and 2, 4 and 1’ she points at each number in the V as she reads out the first recorded outcome, ‘and 1,5,3..2,4,3’ as she points to the recording of the second V |
| TC1:5 Teacher: ‘but you’ve said its not possible to have 2 at the bottom’ |
| TC1:6 Both: ‘yeah’ |
| TC1:7 Teacher: ‘so what is possible at the bottom?’ |
| TC1:8 Mia: ‘1 and 3 so far’ |

Here, the teacher (turn TC1.3) builds on Ruby’s identification of impossibility in turn TC1:2, by asking what is possible. However, Mia’s response (turn TC1:4) does not build on Ruby’s earlier comment but results in a listing of all the numbers in the completed Vs. Mia’s initial partial structural awareness noted in 4.1.1 is further evidenced in her listing of the numbers in each arm at this point, which meant she included the bottom number each time, thus saying it twice. The teacher’s attempt to refocus attention on the bottom number in turns 5 and 7 results in Mia listing the bottom numbers so far found to be possible, but not the property that connects them.
In contrast, two examples of more productive teacher intervention given earlier in this section are re-presented here:

**Teacher 2:** ‘What kind ..not necessarily the value of the numbers but the kind of numbers I’ve left you with.’

Extract 4-31

**Teacher 1:** ‘look at the totals and see what they have in common’

Extract 4-32

A third example provides affordance for the shift required for Rose and Hannah building on the recording above:

**Teacher 2:** ‘Is there anything the same about all of the numbers?’

Extract 4-33

In each of the above the teacher’s question provided greater clarity of features of the results that were of significance (Lobato et al., 2013; Voutsina and Ismail, 2011). In the first, the teacher directs attention away from the surface property (Bergqvist and Lithner, 2012) of number size; in the second and third the teachers’ questions successfully steers pupil attention away from considering properties of individual numbers towards considering a property of the set.

Secondly, data suggests that the mode of representation (Bruner, 1966) may have been influential. Despite the emphasis and frequent encouragement to use representations enactively that was part of the intervention, pupils’ outcomes were recorded on sheets symbolically. Thus, teacher interventions that invited pupils to look at their sheets were inviting engagement with the symbolic mode. Research has however suggested that the use of concrete, or enactive, representations are more supportive for children in noticing features and in constructing reasoning (Marley and Carbonneau, 2014; Fyfe et al., 2014; Houssart, 2004). Indeed data presented in 4.2.1 would suggest that Daisy’s engagement with Numicon provided a productive affordance in the development of her awareness that the total of the set of numbers was an odd number. The provision by her teacher of just five Numicon pieces representing each of the numbers 1-5 was an attempt to provoke awareness of the odd/even property of the remaining four numbers in the set when a base number is removed. For example, removing a base number of 2, leaves 1,3,5 and 4; three odd
numbers and one even. It is impossible to construct two equal pairs from these four numbers. This was the only occasion in which a class teacher strongly promoted the use of enactive representation to support pupils to examine features of their recorded results and represents the only occasion in research lesson 1 in which the lower attainer demonstrated a mathematical property awareness in advance of their higher attaining partner.

At earlier stages of this lesson, the representation of number provided by Numicon might also have supported earlier awareness by lower attainers of the odd number property connecting the successful base numbers of 1, 3, 5, shown grouped together at the top of Figure 4-21. The visual image may have drawn attention to the sameness of these three shapes and to the difference between these and the shape of the 2 and 4 shown at the bottom of the same image.

![Figure 0-21 Numicon representation of numbers 1-5](image)

Lastly, in research lesson 1 in particular, the challenge in provoking pupils’ awareness of this property appeared connected, for two of the pairs, to limitations in subsequent reasoning based on this property and in their use of it in the next stage of the task. Joe and his partner Ash continued to pay scant attention to the possible base numbers even after this had been drawn out in the class plenary. Mia and her partner Ruby showed no interest in pursuing an explanation of why only odd numbers could be positioned at the base. This suggests that pupils were not able to make the most productive shift in the way in which they attended to the
mathematical properties concerned; a shift from attending to individual properties to attending to relationship between properties (Mason, 2011; 2003). Mason (2008a) argues that reasoning requires that pupils recognise that a property can be used as a basis for reasoning and in this instance this awareness may not have been present; quite simply the property of the base number was viewed as an end in itself. Whilst this outcome has been noted in the case of lower attainers (Mulligan, 2011), data from research lesson 1 suggests that this applied to both higher and lower attainers alike.
Chapter 5: The impact of the focus on noticing on authority relations

The previous chapter addressed the mathematical awarenesses demonstrated by lower attainers and the impact that these awarenesses had on task progress in the preliminary and research lessons. In this chapter I address the third and fourth research questions. Findings associated with each of these questions are summarised below:

3. Research question 3: ‘In what ways does a pedagogical focus on noticing influence the nature of the interaction between lower attaining pupils and their higher attaining partners?’
   d) A focus on noticing was associated with more fluid authority relations between the focus lower attaining pupils and their partners

4. Research question 4: ‘In what ways does a pedagogical focus on noticing influence the nature of the expectations for engagement and activity conveyed through class teachers’ introductions and questions?’
   e) Class teachers’ introductions and questions in the preliminary lesson reflected a focus on following the rules of the task and ensuring accuracy
   f) Class teachers’ introductions and questions in the research lessons reflect authority distributed to pupils to make decisions and establish understanding jointly

In relation to research question 3, finding (d), I focus on the nature of interaction between the pupils in each pair and present analysis of selected examples illustrative of the finding expressed above. Analysis has drawn on the same critical sequences in which mathematical awarenesses were demonstrated and begins with the awarenesses that were expressed. Thus, in this chapter I revisit data extracts presented in the preceding chapter. Through the lens of authority relations, expressed awarenesses are interpreted as claims to expertise on the part of the lower attainer. Immediately subsequent interaction and activity is examined to evaluate the extent to which these claims are considered to have intellectual merit.
and to explore their influence on the progress of the task. Analysis considers the three connected elements of perceived intellectual merit, spatial privilege (Dookie and Esmonde, 2012; Engle et al., 2014) and access to the activity space (see 2.3.5) together in the examples presented.

In section 5.1, I present analysis from the preliminary lesson, focusing on higher attainers’ responses to lower attainers’ process and structural awarenesses. Discussion of higher attainers’ responses to selected examples of all categories of awareness evident in the research lessons follows in section 5.2.

5.1 Authority and influence in the preliminary lesson

5.1.1 Distribution of authority

In the preliminary lesson in each classroom, the teacher used a set of large foam dice to model the construction of the dice train and to explain the two rules that had to be satisfied for a train to be valid. The following is typical of the reminder given at the end of this modelling:

Teacher 2: You’ve got to make sure you follow both rules.

Extract 5-1

Pupils were also instructed to calculate and record the ‘spot total’ (see section 3.6.5); pupils were asked to establish their own ways of calculating this quantity but were given precise instructions as to the dice faces that needed to be included.

Following the clarification of the rules the instruction given the goal of the task was clarified:

Teacher 1: So this is your challenge. How many different dice trains can you make following the rules. How many? Find one? good effort, find more
Teacher 1: So you make a dice train that works, following the rules, work out the spot total.
Teacher 3: I want to know how many different trains you can make and what their spot totals are

Extract 5-2
Interventions at pupils’ tables typically focused on accuracy and completion. For example:

**Teacher 1:** What will I see when I pull these two carriages apart?

*Extract 5-3*

This question in Extract 5-3 tests whether the train created met rule 1 of the task. The teacher continued to check that the train met both rules and then congratulated the pupils and left the table.

**Teacher 2:** How can you be sure that you included all the dice faces? *(in the spot total).*

*Extract 5-4*

This question (Extract 5-4) focuses on pupils’ approach to calculation and usually involved a check of accuracy of addition as well as requiring description of the way in which pupils organised the count. In this class the teacher stopped the class to ask them to reflect on their spot total calculation saying that too many errors had been noted. In the course of this exchange the teacher also asked pupils how they would know if they had found all the solutions, but this was not pursued at the time or later.

In school 3 where the focus pupils struggled to manipulate the dice to successfully make valid trains, teacher interventions at tables focused on recapping the rules and modelling the fixing of one face and rotating one die to consider the possible top faces that might arise. This was focused on solution finding only and did not pursue impossible or possible arrangements.

Such interventions, together with the earlier task instructions clarified that in the context of the lesson, what ‘counted’ mathematically (Oyler, 1996) was the finding and recording of multiple correct solutions. In terms of the expectations, obligations and entitlements (Gresalfi and Cobb, 2006) that clarify the way in which authority was distributed in this lesson, pupils were *expected* to work towards the task goal of finding correct solutions. They were *entitled* to establish their own ways of calculating and recording their calculations. An *obligation* was placed on them to evaluate the legitimacy of their dice train solutions and the accuracy of their calculations (Gresalfi and Cobb, 2006; Amit and Fried, 2005).
5.1.2 Response to awareness of mathematical process

The single example of mathematical process awareness demonstrated in this lesson came from lower attainer Mia working with higher attainer Ruby, reported in 4.1.4. Below, this same process suggestion is re-presented (turn 1:2 below) and I draw on the preceding and subsequent interaction to explain why this suggestion did not influence subsequent activity. The exchange in question took place early in the first sequence of pair activity, between lower attainer Mia and her higher attaining partner Ruby:

Mia and Ruby are working with four dice; three are pink and one is orange

1:1 Ruby picks up pencil, taps it on table, gazes to right (away from Mia), and then ahead. Meanwhile Mia has all three pink dice in front of her

1:2 Mia: ‘Shall we start with ones?’ Mia orients all three pink dice in line with rule 1, so that ones are on top. She glances towards Ruby who is still staring ahead but who then returns her focus to the activity, and turns towards Mia.

1:3 Ruby immediately picks up all three of the dice Mia had been working with and moves them to be squarely in front of her: ‘Right, I didn’t see what you done there, so’.

1:4 Mia is holding the remaining (orange) die and makes no response. She looks at this die closely, rotating it slowly. She places two fingers on one face then turns the die to identify the number on the opposite face.

1:5 Ruby: (tilting joined dice so she can see adjoining faces) ‘four and four, three and three’.

1:6 Mia puts down the orange die and stands up leaning towards Ruby and the dice arrangement.

1:7 Ruby picks up the orange die, placing it on the train in line with rule 1. Its top face is six. Ruby turns to Mia, smiling: ‘six and two doesn’t equal’.

Extract 5-5

In Extract 5-5 above, Mia’s expressed awareness (turn 1:2) of a mathematical process also represents a claim to mathematical expertise; she suggests a way to proceed with the investigation. Her utterance (turn 1:2) demonstrates that she is intent on being part of the decision making about processes (Boaler and Sengupta-Irving, 2016; Gresalfi and Cobb, 2006) in conjunction with her partner; her use of a questioning structure and glance towards her partner invites this discussion.

Ruby does not afford Mia spatial privilege (Engle et al., 2014) at the outset of the exchange. She is oriented away from the activity and her gaze is elsewhere (turn 1:2). Ruby’s response (turn 1:3) does not address Mia’s suggestion, this ignoring of her
contribution resonant of existing research findings (Wood and Kalinec, 2012) and an evaluation that the idea has limited intellectual merit. This is compounded by Ruby’s counter claim to mathematical expertise authority in asserting a right to interrupt Mia’s activity to check the accuracy of Mia’s partial solution (Esmonde, 2009b) against rule 1 of the task. This physical intervention limits Mia’s access to the activity space in this section of the activity and serves to illustrate the interconnected nature of these two elements of the authority relations between the pair. Ruby did not return the dice to Mia once the accuracy check was made; nor did Mia request that she did so. The physical intervention thus enabled Ruby to assume control of the resource and the next stages of the investigation (Dookie and Esmonde, 2012).

Mia’s lack of response or challenge to Ruby’s action illustrates the role of both parties in forming such relations (Gerson and Bateman, 2010). Whilst this may be due to her confidence that rule 1 had been satisfied, she nevertheless made no protest at the interruption to her activity, the removal of the dice, or to the implied assumption that she may have failed to meet the rule. Whilst Ruby is engaging in her accuracy check, Mia engages in further exploration of the remaining die. This evidences her mathematical engagement and suggests an interest in understanding the structure of the resource. Having completed the accuracy check (turn 1:5), Ruby completes the arrangement Mia had started by positioning the orange die to form a funnel (Figure 5-1). This resulted in an inaccurate train; it showed 6 on the top of the funnel with the carriage tops summing to 2.
Ruby’s tone as she turns to Mia and says (turn 1:7) ‘six and two doesn’t equal’ is dismissive; together with her slightly superior smile this conveys that she considers Mia’s suggested approach as foolish and having no intellectual merit. Without further discussion this systematic specialising (Mason et al., 2010) approach is dropped. Had the pair pursued the systematic specialising suggestion, despite the first attempt being unsuccessful, they may have been able to engage in consideration of increasingly large carriage dice totals in an ordered way. This could have paved the way for reasoning about possibilities.

As was noted this was the only demonstration of awareness of an applicable mathematical process beyond the use of random specialising in the preliminary lesson. However, analysis of the exchange is included because it is instructive in identifying differences in the nature of interaction relating to mathematical processes between the preliminary and research lessons.

This exchange between Mia and Ruby reveals a feature of the higher attainers’ activity that was common across three of the four focus pairs in this preliminary lesson. Here I refer to the physical intervention by the higher attainer to remove the dice from the lower attainer ostensibly to check the accuracy (Esmonde, 2009b) against rule 1 of the task. This restriction of lower attainers’ access to the activity space was almost always unchallenged by the lower attainers, signalling their acceptance of the expertise authority of their partner in this regard (Esmonde, 2009b; DeJarnette and Gonzalez, 2015).

In addition to the example from Mia and Ruby reported above, in the partnership between higher attainer Ash and lower attainer Joe, Ash’s influence over Joe’s access to the activity space was frequently manifested in him holding the dice and releasing them to Joe when the latter requested a turn, but then removing them part way through Joe’s trial or joining in with manipulation prior to so doing. Interaction between this pair was combative but jocular; Joe maintained his involvement throughout, despite Ash frequently asserting control. The following extract is taken from mid-way through the lesson and in an instance in which Joe has been manipulating dice. Two dice are joined, and he pauses to reach for a third:
As Joe pulls his hands away from the two joined dice Ash leans in and takes them both:

3:6 Joe: ‘what you doing?’
3:7 Ash: ‘I just wanted to check, 4 and 4’ He pulls in a third die, matching it and with 2 on top face.

Extract 5-6

Joe’s protest in Extract 5-6, turn 3:6, prompts an explanation from Ash, but Ash continues his intervention nevertheless. His continuation of the train attempt that Joe has started denies Joe the opportunity to pursue his thinking; this attempt fails and with no discussion of it. Such instances were common between this pair.

A second and more troubling example is provided by higher attainer Poppy and her lower attaining partner Rose. Data presented in 4.1.4 reflects an awareness from Rose interpreted as structural; the dialogue below immediately precedes the expression of this awareness:

3:1 Poppy puts down her pencil and turns to Rose: ‘I know how to do it’. She turns her head away but at the same time takes hold of all dice and removes them from Rose’s hands. Rose allows this to happen and turns towards Poppy.
3:2 Poppy picks up all dice in her left hand, clutching them to her. Her pencil is in her right hand: ‘right, what number do you want?’ Rose makes no comment. Poppy records something in her book then turns back to Rose: ‘you can’t copy’ and shuts her book.
3:3 Poppy selects the green die and says: ‘right 6’ and turns this over.
3:5 Poppy places the funnel die on top of another die. 6 is showing on the top but the arrangement is not in line with rule 1 at this stage. Rose says something inaudible to Poppy who replies: ‘you have to get on with your own work, you don’t have to wait for other people’
3:6 Rose turns to Poppy with sulky expression. She flops her book down. Poppy says: ‘I’m trying to help you but if you don’t want me to I won’t.’ She drops all dice on the table.

Extract 5-7

It is not clear on what basis Poppy makes the expertise authority claim in Extract 5-7, turn 3:1, given her limited engagement to that point and the lack of spatial privilege she has afforded to Rose and the activity in which she has been engaged. In turn 3:1 she asserts control of Rose’s access to the activity space by removing the dice from her and (turn 3:2) makes a claim to mathematical expertise in the task by suggesting that she can make any solution requested. Her secretive follow up of hidden recording in the same turn is a further claim of superiority and indicates that this will be an individual solution and not a collaborative endeavour. The reason for this is
maybe indicated in the rebuke in turn 3:5 in which Poppy appears to suggest that Rose has not been making productive use of her time up until this point. Video evidence would suggest that the reverse is true. Finally, her construction of herself as the helper with superior knowledge in turn 3:6 is accompanied by a threat of removing the assistance offered should Rose not behave appropriately. This entire exchange constructs Poppy with an intellectual authority that data indicates is not warranted (Engle et al., 2014; Langer-Osuna, 2016). The use, by Poppy, of ‘identifying’ talk (Wood and Kalinec, 2012), focusing on Rose’s perceived behavioural shortcomings further constructs Poppy as the senior partner with the right to admonish.

5.1.3 Response to awareness of structure and generality

Two of the four lower attainers (Joe and Mia) demonstrated awareness of structure in their respective preliminary lessons. In the case of lower attainer Joe, working with higher attainer Ash, this was reported in 4.1.4 and is re-presented here:

<table>
<thead>
<tr>
<th>Joe holds two dice in his hand and is beginning a trial</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:3 Joe: ‘Well you can’t make 7 so,’</td>
</tr>
<tr>
<td>1:4 Ash: ‘yeah so’</td>
</tr>
<tr>
<td>1:5 Joe: ‘So let’s try 3 and 2’. He places the two dice on the table, 2 and 3 uppermost</td>
</tr>
</tbody>
</table>

In Extract 5-8, Joe’s expressed structural awareness constitutes a claim to expertise authority in the context of the task. Presented as a logical construction across turns 1:3 and 1:5, he claims expertise in responding to the constraint he has identified (turn 1:3) through the suggested use of specialising (Mason et al., 2010) with two numbers for which he is aware the total is less than 7. Whilst Ash’s utterance in turn 1:4 appears to afford this idea merit, the tone in which it is delivered shows this not to be the case. The tone is sarcastic; it is in effect a counter claim to expertise and infers that Ash is already aware of this same constraint. Thus, Ash dismisses the intellectual merit of the claim (Langer-Osuna, 2016), not on the basis of its content, but on the basis that it is old news, and hence not worthy of being made.
Joe makes the same claim to expertise authority several minutes later. At this point Ash is placing the funnel on to the train such that the carriages sum to 10 (Figure 5-2).

![Figure 0-2 Ash works on an impossible arrangement](image)

It is the impossibility of this arrangement that Joe references in his comment noted in 4.1.4 (Extract 4-16): ‘How are we supposed to make 10, there’s no 10 on the dice’. On this occasion the claim to expertise is expressed as a legitimate challenge to Ash’s current activity: Ash’s actions are not in line with the counter claim and dismissal of turn 1:4 in Extract 5-8 above. Ash’s control of the dice affords him spatial privilege (Engle et al., 2014) and Joe orients himself towards the dice and Ash’s activity. Joe does not physically intervene, preferring a verbal argument of his case, this time accepted by Ash as valid. It is notable, in line with other research (M. Barnes, 2005) that the lower attainer needed to make the claim twice before an acknowledgement by the higher attainer of the validity of his lower attaining partner’s contribution was forthcoming.

A similar exchange noted between lower attainer Mia and higher attainer Ruby on this same topic was also reported in 4.1.4 and re-presented here:

<table>
<thead>
<tr>
<th>Extract 5-9</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4:1</strong> Mia, looking towards the train: ‘it can’t be more than six though because these dices (touching the FD) don’t go more than six</td>
</tr>
<tr>
<td><strong>4:2</strong> Ruby, still with head down: ‘I know’. She continues to rotate a carriage die, groaning when the face turned to the top is a six</td>
</tr>
</tbody>
</table>
Mia’s claim to expertise (Extract 5-9 turn 4:1) relating to this same structural constraint is expressed as she leans over to look at an arrangement that Ruby is working on. As Figure 5-3 shows, at this point Ruby (on the left of the image) had constructed the funnel and carriage beneath and was holding these in her left hand.

The first carriage also appears fixed and she is carefully rotating the second carriage. As with the case of Joe above, Mia’s claim to expertise is a challenge to her partner’s activity. Here it was met with a limited response; a mere ‘I know’ from Ruby. However, her current activity does not bear out this claim to the same expertise and Mia’s challenge is valid. Whatever the eventual top face of the die Ruby is rotating in the image above, since the first carriage (5) already equals the number on the raised funnel die (5), the arrangement is impossible at this point. The activity and position of both girls as shown in Figure 5-3 echoes earlier research findings: Ruby, on the left of the image, has control of the dice and this control enables her to avoid explaining the apparent contradiction in her actions and her speech (Dookie and Esmonde, 2012). This same control means that she is afforded spatial privilege (Engle et al., 2014) unlike Mia, who is not attended to when speaking and has to lean across to remain connected with the activity underway.

Separately and in their respective lessons, lower attainers Daisy and Mia both noted a further constraint relating to numbers on opposite faces of the die. These have both been presented in 4.1.4 and are not re-presented here; of note is that in neither case
did their higher attaining partners respond verbally or in action (Wood and Kalinec, 2012; Esmonde, 2009b) to these claims to expertise. In both cases, higher attainers were engaged in dice manipulation when the awareness was expressed and continued in this activity without comment or physical acknowledgement; control of the resource again removing the need for communication or justification of action (Dookie and Esmonde, 2012). Thus in both cases the intellectual merit of the claim that the awareness represented was dismissed by being ignored.

The provision of minimal responses by the higher attainer to lower attainers’ suggestions and ideas has been noted in earlier research (Wood and Kalinec, 2012; Esmonde, 2009b) as has the tendency to dismiss the value of their partner’s contributions regardless of their merit (Langer-Osuna, 2016; Engle et al., 2014). Indeed, the exchanges reported above perhaps reflect a difference between the higher and lower attainers in terms of communicative intent which also mirrors earlier research findings (Sfard and Kieran, 2001). Lower attainers’ comments typically display an initiating attitude, signalling interest in communication about the task, and attempting to establish a joint approach to it. Higher attainers tend to be both un-initiating of discussion and unresponsive to it (Sfard and Kieran, 2001, p62); their lack of verbal communication signalling a focus on object level issues, i.e. finding solutions. For all pairs these minimal or dismissive responses on the part of the higher attainer arguably represent missed opportunities for co-construction of reasoning (Mueller et al., 2012) about possibilities that may have enabled further progress in the task beyond solution finding.

The examples presented thus far in this section echo much of existing literature. In particular, the mediating role of resources which enables higher attainers to continue to act without justifying their actions is in line with the findings of Dookie and Esmonde (2012). The dismissal of lower attainers’ claims to expertise and the limited response to their ideas has also been established (Wood and Kalinec, 2012; Esmonde, 2009b). As a result the tendency in this lesson was that the higher attainer had greater opportunity to engage mathematically (Esmonde et al., 2009; Esmonde, 2009b; Cohen and Lotan, 1997; 1995; Esmonde and Langer-Osuna, 2013). As Table 5-1 shows, higher attainers responses closed down opportunities through dismissal,
minimal responses, physical intervention and the making of counter claims meaning that lower attainers ideas were not afforded opportunity to develop traction, with resultant minimal impact on task progress.

<table>
<thead>
<tr>
<th>Authority and influence analytic focus</th>
<th>Preliminary lesson summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>The perceived intellectual merit of lower attainers’ ideas</td>
<td>Ideas evaluated as having little merit, tending to be ignored or dismissed with minimal responses</td>
</tr>
<tr>
<td>Lower attainers’ access to the activity space</td>
<td>Frequently access to resources interrupted or removed by higher attainer; visual access sometimes limited</td>
</tr>
<tr>
<td>The degree of spatial privilege afforded to lower attainers</td>
<td>Lower attainers were usually not afforded spatial privilege, were not physically attended to when speaking or engaged in activity</td>
</tr>
</tbody>
</table>

*Table 5-1 Authority and influence in the preliminary lesson*

Lower attainer Daisy and higher attainer Alice to an extent present a counter example to the other three pairs. Here the continued participation of both pupils suggests a more shared authority (DeJarnette and Gonzalez, 2015; Turner et al., 2011); typically, one pupil would make a claim to expertise by asserting an idea and the other conceded the dice for the idea to be explored thus reflecting mutual acceptance of the merit of each other’s ideas and enabling continued shared access to the activity space. Unlike in other pairs, the persisting involvement of both pupils in dice manipulation enabled both to become adept at solution finding. As noted in 4.2.3 this extended to the pair identifying a target funnel total and then manipulating the dice to create the train to match. To this extent it can be argued that their greater co-operation supported progress in solution finding; this pair found more solutions than all others. However, as with other focus pairs, Alice’s responses to Daisy’s structural awarenesses (4.1.4) were limited and this restricted the extent to which Daisy’s structural awarenesses were enabled to impact productively on their progress in reasoning about the solutions found.
5.2 Authority and influence in the research lessons

Analysis of the nature of interaction between the focus lower attainers and their partners through the lens of authority relations provides insight into how lower attainers’ awarenesses had greater influence on task progress in the research lessons than had been the case in the preliminary lesson. Similarities exist across different pupil pairs and different lessons; selected examples are drawn out for detailed analysis with shorter examples provided to illustrate pertinent differences where these exist. As in the previous section, analysis begins with consideration of the distribution of authority evident in the teachers’ introductions and interventions.

5.2.1 Distribution of authority

Across all classes and all lessons teacher introductions highlighted the value attached to noticing and to sharing this noticing. The following examples are representative of teacher introductions:

<table>
<thead>
<tr>
<th>Extract 5-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher 1 (research lesson 1): ‘Today’s real focus is on what you notice in your mathematics. So a lot of what you do in terms of the calculating won’t be terribly complicated.. it’s about what you notice, what you see, what you think. A lot of the emphasis is going to be on writing that down and sharing that.’</td>
</tr>
<tr>
<td>Teacher 2 (research lesson 1): ‘OK guys. We’re thinking about noticing and the main thing we are looking for today is that you articulate, which means, say, say out loud and have a discussion about, the things that you notice. Sometimes,...we keep things in our head, but I’m saying share what you’ve seen, even if you’re not sure why, if you’ve noticed something, share it. Because it might be that the person next to you goes ‘Ah’ and goes with it. If you don’t say it, you’re not going to get that are you? So even if you’re not sure why, if you’ve noticed something we’re expecting to hear it being discussed today to see if it’s useful or not. And then hopefully, that noticing is going to help you use your reasoning’</td>
</tr>
</tbody>
</table>

The central focus of noticing is further underlined (Extract 5-10) by teacher 1 who expressly identifies that the numbers and calculations involved were not the challenging part of the lesson; the emphasis was on noticing. Teacher 2 stresses the importance of sharing noticing regardless of whether its usefulness has been established. Here the potential for productive dialogue when noticing is shared is the key emphasis. Further stressing of noticing was typically conveyed as teachers
introduced the tasks. Here again the teacher’s question shows that the emphasis is not on the sum of the numbers but on what pupils notice about the sum:

<table>
<thead>
<tr>
<th>Extract 5-11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher 3 (research lesson 3): ‘If I take any three numbers from any of the bags and add them up, what do you notice? Talk to your partner’</td>
</tr>
</tbody>
</table>

Introductions to the tasks frequently included a short activity of the kind referenced in Extract 5-11. Initial noticings were discussed; this served to model the behaviour and the class teacher’s interest in both noticing and discussing noticings. Interventions at tables typically began with the teacher asking the pupil pair what they had noticed and what they thought this meant in the context of the task. Recording sheets invited pupils to record what was noticed and they were frequently encouraged to use this section of their recording sheets. Mid and end of lesson plenary sessions typically began with an open question about what had been noticed; pupils’ responses informing subsequent questions:

<table>
<thead>
<tr>
<th>Extract 5-12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher 3 (research lesson 2)</td>
</tr>
<tr>
<td>‘OK so what have you noticed? What works and what doesn’t?’</td>
</tr>
<tr>
<td>‘So what is the general rule about finding two numbers that can be put together?’</td>
</tr>
</tbody>
</table>

In Extract 5-12 above the first opening question resulted in a range of specific examples of successful pairings. The teacher probed for awareness of the generality that pairs had to be multiple/factor pairs for there to be no remainder.

As well as discussing what had been noticed, teacher interventions probed for what was possible:

<table>
<thead>
<tr>
<th>Extract 5-13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher 3 (research lesson 2): ‘Can 1 go into anything else?’</td>
</tr>
</tbody>
</table>

The question in Extract 5-13 exemplifies the expectation that pupils will explore and think about possible pairings of the cards. Note that the accuracy of current pairings is not the teacher’s concern, rather that pupils consider all the possibilities that a particular card presents.
Thus in the research lessons, teacher introductions and questions exemplify a distribution of authority (Gresalfi and Cobb, 2006) that shifts the emphasis from solution finding to noticing. Specifically, teachers articulated an expectation that pupils would engage with the task and work towards the task goal of noticing. These introductions further placed an obligation on each pupil to discuss what they had noticed with their partner. The entitlement for each pupil arises from the expectation and obligation: that each pupil was entitled to share their noticings with their partner and further that they were entitled to be listened to.

5.2.2 Research lesson 1: response to awareness of process

In research lesson 1, (see 4.1.1) Mia makes the same mathematical process suggestion that she made at the outset of the preliminary lesson:

<table>
<thead>
<tr>
<th>1:3 Mia: ‘Let’s try first with 1 at the bottom’</th>
</tr>
</thead>
</table>

Extract 5-14

In a contrast to the preliminary lesson, this suggestion had influence on the development of the task. On this occasion her vocalised claim to mathematics expertise in the form of the process suggestion, was immediately followed by action in bringing the 1 card into position and fixing it in this position with her left hand (Figure 5-4).

![Figure 0-4 Mia secures the position of the 1 card](image)

This served to reinforce the claim to mathematics expertise, a claim further emphasised by the absence of insistence on the position of any other card. Thus Mia
is not attempting to dominate the exploration but emphasises a very particular requirement. Figure 5-4 also shows that both girls have access to the activity space; the cards are between them in sight and reach of both (Dookie and Esmonde, 2012). Moreover they afford each other spatial privilege: both are oriented slightly inwards towards each other and the card set. That Mia’s continued hold of the 1 card in position enabled trials to proceed with this variable fixed for long enough for a solution to be found, validated her idea and affords it intellectual merit, arguably increasing her intellectual authority as a result (Engle et al., 2014).

Subsequent to this first example, the extract below provides further insight into the impact of Mia’s continued access to the activity space. This extract is taken from shortly after the first solution was recorded:

**Extract 5-15**

*The cards remain in the position of the solution just recorded. Both girls are ready to begin a new trial:*

2:1 Mia: ‘Shall we change..?’ Mia moves the 5 from its position in the middle of the right hand arm to place it in position to be the bottom number. Ruby takes the 5 card back from Mia and slides it to her left slightly.

2:2 Mia moves to retrieve the 5 with left hand: ‘Right so that’s at the bottom’. Ruby pulls it away sharply to her left

2:3 Ruby: ‘wait wait wait. I think we should have.. I think we can make more that have, with 1 at the bottom’ Ruby re-constructs the V they have just recorded as she speaks. ‘I think there has to be more of them, seriously... So maybe we can switch’

Extract 5-15 begins with Mia claiming mathematical expertise in making a process suggestion and accompanying it with action (turn 2:1). She is not dissuaded by Ruby’s non-verbal challenge (turn 2:1) in interrupting her activity and makes the same move in turn 2:2. This greater insistence from Mia prompts Ruby to provide a reason for her objection and counter claim (Webb et al., 2014). It must be acknowledged that Ruby’s objection is reasonable; at this point they have not established conclusively that there is only one solution for each base number. However, this represents a clear difference from the preliminary lesson where control of the resource enabled the higher attainer to avoid explanation or justification (Dookie and Esmonde, 2012). In this instance shared access to the activity space and its resources enables both to engage mathematically.
5.2.3 Research lesson 1: response to awareness of property

The mathematical property awareness expressed by Daisy and reported in 4.2.1, was the first clear indication of her influence on the task. Daisy’s partner Jess had led manipulation and had been more successful in finding solutions. Daisy had been involved, but aside from her occasional random specialising (Mason et al., 2010) with the card set, her involvement had been to approve decisions that Jess suggested and to collaborate in checking solutions. Thus it was important that when she had a clear contribution to make, it had impact. Her awareness is re-presented below together with the immediately subsequent response from Jess:

3:25 Daisy: ‘add all of these up together’. She has her head down and lifts and firmly replaces her stack of Numicon pieces back down on the table; ‘all of the numbers that we’ve got’

3:26 Jess turns back towards the cards on the table, pulling each one towards her as she adds ‘4..9..10.13’ she turns back towards Daisy as she emphasises 13. Daisy still has her head down

3:27 Daisy lets go of her stack of Numicon and turns slowly towards Jess, lifting and pointing her left index finger as she speaks: ‘13 is odd….. so you can’t split it up, like you can with 12 and 14’

3:28 Jess: ‘Oh yeah, oh yeah’ she smiles as she turns her head to her sheet and begins to record. The class teacher stops the class.

In Extract 5-16 Daisy’s claim to mathematics expertise is articulated in 3:25 as an instruction to her partner. Her gesture of lifting then firmly replacing the Numicon on the table (turn 3:25) gives gravitas to this request, results in spatial privilege afforded to her as Jess turns towards her and signals her continued access to the activity space. Her claim is accepted by her partner who calculates as requested and then turns back to Daisy with her response (turn 3:26). Daisy’s reasoning, when articulated in turn 3:27, is slow, careful and accompanied by a pointing gesture which serves to underline her confidence in her awareness. Here the lower attainer’s hold of the resource which the pair have recently been using is not utilised to bypass justification or explanation (Dookie and Esmonde, 2012), but to affirm the significance of an emerging awareness. The continued spatial privilege (Engle et al., 2014) and access to the activity space that Daisy’s tone and actions generate enables
her to continue her line of reasoning to its conclusion. Her partner’s pleasure at the awareness is evident in turn 3:28.

5.2.4 Research lesson 1: response to awareness of structure

In research lesson 1, two focus lower attainers, Joe and Mia, working with higher attainers Ash and Ruby respectively, had expressed structural awarenesses that impacted positively on task progress. The development of influence reflected is similar in both cases.

Joe’s emerging structural awareness, presented in 4.1.1 and re-presented below, shows Joe drawing on the need for equivalence in arm pair totals to persuade Ash that a solution is incorrect:

2:33 Joe: ‘no, yeah 5’ he again indicates the left arm pair (3+2) and then points at the right pair (5+1) ‘and that’s 6 yeah?’

Extract 5-17

Extract 5-17 represents a claim to mathematical expertise through asserting conditions that must be fulfilled for a solution to be viable. Influence is possible for Joe through the spatial privilege afforded to him as he makes his assertion; he is physically attended to and uninterrupted (Engle et al., 2014). This arises partly through his current access to the card set (Dookie and Esmonde, 2012), but also through his approach to convincing his partner. His utterance invites engagement, through its questioning style and its combination of gesture and speech and reflects communicative intent.

In all examples presented here, the higher attainers’ response has differed in important ways from that presented in the preliminary lesson. However, the lower attainers’ actions have also differed. All have found ways to ensure that they were afforded spatial privilege and continued access to the activity space: Daisy, holding the resource spoke with urgency and insistence; Mia used a hand to ensure that the position of a card was maintained; Joe used gesture towards the resource and tone to invite shared understanding.
Indeed the two images below illustrate the difference in spatial privilege frequently afforded to and gained by lower attainers in the preliminary lesson and in research lesson 1. Each image shows the point at which a claim to expertise in the form of an awareness was articulated. In these two images, lower attainer Mia and higher attaining partner Ruby are shown in the preliminary lesson (Figure 5-5) and then in research lesson 1 (Figure 5-6). In research lesson 1, Mia on the left of the image is afforded spatial privilege; Ruby is oriented towards her and observes Mia’s hand gesture as she indicates two arm pairs of the V shape. Both girls have visual access to the resource; Mia’s closer physical access and opportunity to use the resource to support her articulation contributes to the influence that she has at this point. In the preliminary lesson Ruby on the left of Figure 5-5 did not afford Mia the same spatial privilege, keeping her head down as Mia spoke and limiting Mia’s access to the activity space. It required some effort to lean over and peer at the action that Ruby was undertaking in a solitary manner. Engle et al (2014) consider that spatial privilege is a contributory, indirect factor in the construction of influence and the images above bear this out. The opportunity to be attended to when articulating an awareness enabled the intellectual validity (Engle et al., 2014) of the awareness to be accepted in Figure 5-6. Access to the activity space however, I would argue affords direct influence on the task.

5.2.5 Research lesson 3: response to awareness of structure

Examples from the final research lesson again reflect a greater degree of spatial privilege afforded to lower attainers together with positive responses from higher attainers at the awarenesses expressed. Examples from Rose, Mia and Joe in their
respective lessons have been presented in 4.1.3; the interaction between lower attainer Rose and her partner Hannah is re-presented below:

3:8 Rose jumps in quickly as Hannah finishes: ‘no, no, no, if it was one less it would be zero and there would be nothing there. And that would be a multiple of 3. Zero, three, six..’ she waves hand right and left as she indicates this as part of a times table rhythmic count.

3:9 Hannah, excitedly and mouth open wide: ‘oh yeah!.. Oh yeah!’ Rose is smiling very broadly now

Extract 5-18

Hannah’s excitement is evident (Extract 5-18) in her verbal response and in her facial expression (turn 3:9) and this serves to confirm the merit of Rose’s assertion. Both girls share access to the activity space with resources accessible and visible to both. The pair are physically oriented towards each other and thus both are afforded spatial privilege (Engle et al., 2014) during the dialogue. Subsequent to this extract Hannah articulates how the new awareness enables all numbers in the set to be considered in the same way showing that she built Rose’s awareness into her own thinking, a co-construction of reasoning (Mueller et al., 2012) a further acknowledgement of the intellectual validity (Engle et al., 2014) of Rose’s awareness.

In the parallel lesson, lower attainer Mia’s partner Ruby expressed a similar pleasure of realisation as Mia articulated the relationship to the three times table of the numbers in the set. Her smile and her long exhaled sigh reported in 4.1.3, Extract 4-11, showed that this was indeed, an ‘aha’ moment (Mason et al., 2010) which Mia had provided. Again, access to the activity space was shared and the girls were oriented towards each other affording both spatial privilege such that as Mia spoke and indicated the resource, Ruby attended visually to both speech and gesture.

In addition to the positive response to lower attainers’ expressed awarenesses, a further notable feature from the research lessons is the way in which higher attainers responded to challenge from their lower attaining partners. Here the strongest example comes from Joe and his partner Ash in research lesson 3, though a comparable example exists between Mia and Ruby in research lesson 2. In the extract below (Extract 5-19), re-presented from 4.1.3, Ash shares a conjecture with Joe:
Joe turns to look towards Ash as Ash speaks (turn 1.19) and Ash makes clear attempts to explain his thinking to Joe (turn 1.20 and 1.22). Joe’s close attention to the claim that Ash is making is evident in his response shortly afterwards (Extract 5-20):

2:4 Joe leans towards Ash, who is still writing, head down, and speaks, looking at him: ‘Ah but Ash if you were only allowed to choose 2 it would be in the one times table only’. Ash stops writing and sits up

In Extract 5-20 Ash gives the same close attention to Joe’s reasoning, evidenced through his pause in recording as Joe speaks. The claim to expertise reflected in Joe’s utterance in turn 2:4 is accepted by Ash and given validity through his pause and attention. Below (Extract 5-21) Ash tests out Joe’s idea below (turn 2.9) by selecting two numbers. His acceptance of the validity of Joe’s reasoning is reflected in his exhalation in turn 2.11.

2:9: Ash: ‘4+7’ He looks towards Joe
2:10 Joe: ‘4+7=11’
2:11 Ash exhales, looks pensive and turns towards the board. Joe maintains his focus on Ash, frowning: ‘and what calculation goes into 11, only 1, and 11’ Joe looks towards the board.

Several features characterise the exchange between Joe and his partner and mark it as different to challenge in the preliminary lesson. Firstly, both Ash and Joe are afforded spatial privilege (Engle et al., 2014) in being physically attended to as they speak; eye contact was made throughout the exchange. Secondly, both had access to the activity space; here activity was verbal and the utterances of both were not only uninterrupted but given careful consideration. This meant that the idea was developed through dialogue; in contrast challenge in the preliminary lesson was either ignored or needed to be repeated in order for it to be attended to and the
underpinning ideas remained undeveloped. Here also both appear to share the same intent to communicate ideas, a feature of communication more frequently associated with lower attainers in the preliminary lesson.

5.2.6 Research lesson 2: a partial counter example

Whilst examples reflecting a greater sharing of influence are also present in research lesson 2, some features of interaction in this lesson also present something of a counter example to research lessons 1 and 3. In this lesson examples from both lower attainers for whom data is available demonstrate a persisting greater influence on aspects of the task on the part of the higher attainer. Of the three research lessons, lesson 2 arguably offered some measure of advantage to the higher attainer through its focus on division and remainders; those with more rapid recall of multiplication and division facts could pair cards up more swiftly (Boaler and Staples, 2008). In both these lessons this early swiftness and confidence was seen in the higher attainer.

For example, at the beginning of the lesson, through her response to the class teacher’s question, Rose showed that she understood the notion of remainders after division. However, she preferred to make use of Numicon to confirm both quotient and remainder when calculating examples, such as 13 divided by 6, in the class introduction. As the girls began the card pairing task Rose initially wanted to continue to follow the suggestion of the class teacher and use Numicon to test out pairs of cards. This was thwarted by Hannah as the following exchange (Extract 5-22) illustrates:

| 1:7 Rose looks at the cards and asks: ‘what were you going to do, 19 divided by what?..8?’ She stands and reaches for an eight Numicon piece |
| 1:8 Hannah uses her left arm to try to prevent Rose from reaching out to the Numicon: ‘No, 16 divided by 8, 16 divided by 8’ |
| 1:9 Rose continues with her plan and is still standing as she takes an 8 numicon. ‘no no no, it might work though’. Hannah has leaned over to right, elbow on table and leans the side of her head on her hand as she looks left towards Rose |
| 1:10 Hannah: ‘no it doesn’t Rose’ Hannah sits up and holds Rose’s arm to stop her activity. ‘How would that work?’ She leans in and puts her hand on the 8 piece currently covering part of the Numicon representing 19 to enforce the halt and turns towards Rose: ‘You have to know that in your head’ she takes the Numicon 8 from Rose, ‘19 isn’t in the 8 times table so it’s not going to work’ |

Extract 5-22
Hannah’s reasoning has already enabled her to refine her original selection of 19 divided by 8 (turn 1:7) to 16 divided by 8 (turn 1:8) before Rose has been able to select the first Numicon piece. Hannah’s first attempt to prevent the use of Numicon is non-verbal (turn 1:8); Rose’s insistence on her course of action (turn 1:9) provokes both a more insistent physical rebuttal (turn 1:10) together with a reasoned response. The construction of influence through physical intervention in the activity of the lower attainer is reminiscent of the preliminary lesson; here however Hannah is not intervening to check Rose’s work but to change the course of her activity. Demonstrating a different way in which a resource can be used to effect influence than those identified by Dookie and Esmonde (2012), Hannah seeks to prevent use of the resource in order to change the way that Rose engages with the task. Her inclusion of a conceptually underpinned response (Webb and Mastergeorge, 2003) enables Rose to gain awareness of division as connected to multiplication, and points to the use of existing times tables knowledge to identify successful pairs. This explanation was sufficient for Rose to develop the structural understanding that was the foundation for her successful identification of factor/multiple pairs; data presented in 4.1.2 shows Rose gaining in confidence in this as the task proceeds.

Whilst Hannah successfully claimed an intellectual authority (Langer-Osuna, 2016) through this exchange, her care in this and subsequent exchanges in explaining the relevant mathematical ideas shows her aim that Rose could participate meaningfully and with understanding in the task. It was not, as the findings of Esmonde (2009b) and Wood (2013) might suggest, designed to confine Rose’s engagement to the procedural aspects of the task, or to reject the collaboration in favour of an individual focus on the task (Sfard and Kieran, 2001).
5.3 Chapter summary

The examples presented above are representative of exchanges across the research lessons in relation to the three focus analytic areas. In this summary section I consider the differences between the preliminary and research lessons in relation to these three analytic focus areas.

Analysis has focused on very specific aspects of action and interaction and this has enabled me to tease out at times subtle differences between the preliminary and research lessons in how pupils respond and afford each other opportunities to demonstrate and explain their awarenesses. Whilst subtle, these differences have been instrumental in enabling important differences in the extent to which awarenesses of the lower attainers impacted positively and constructively on task progress for the pair.

Table 5-2 summarises the way in which these features are reflected in lesson activity:

<table>
<thead>
<tr>
<th>Authority and influence analytic focus</th>
<th>Preliminary lesson summary</th>
<th>Research lesson summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>The perceived intellectual merit of lower attainers’ ideas</td>
<td>Ideas evaluated as having little merit, tending to be ignored or dismissed with minimal responses</td>
<td>Lower attainers’ ideas evaluated as having merit. Challenge from lower attainers afforded consideration.</td>
</tr>
<tr>
<td>Lower attainers’ access to the activity space</td>
<td>Frequently access to resources interrupted or removed by higher attainer; visual access sometimes limited</td>
<td>Shared access to the activity space; lower attainers’ actions ensure continued visual and physical access to the resources maintained</td>
</tr>
<tr>
<td>The degree of spatial privilege afforded to lower attainers</td>
<td>Lower attainers were usually not afforded spatial privilege, were not physically attended to when speaking or engaged in activity</td>
<td>Pupils usually oriented towards each other. Lower attainers afforded spatial privilege as they speak and use gesture to support their explanations.</td>
</tr>
</tbody>
</table>

Table 5-2 Authority and influence in the research lessons

As the table above shows, a notable feature of the preliminary lesson had been the minimal and at times dismissive responses that lower attainers received when they proposed an idea or expressed a mathematical awareness. Also notable had been the frequent intervention in lower attainers’ activity to check accuracy of their
arrangements. These patterns of interaction constructed an authority relation that positioned the higher attainer as in charge of the task.

Given the authority relations revealed in analysis of the preliminary lesson it is important to stress that, with the exception of the exchange reported between lower attainer Rose and her partner Poppy (5.1.2), data does not indicate any fractious or unpleasant interaction between the pupil pairs of the kind reported elsewhere in literature (Langer-Osuna, 2016; Heyd-Metzuyanim and Sfard, 2012; Wood and Kalinec, 2012). Pair interaction was consistently good natured and friendly; lower attainers accepted the manner of working and did not challenge their partners or demand greater spatial privilege or access to the activity space. Indeed they celebrated the solution finding and owned the work as jointly achieved. This apparent contentment notwithstanding, data reveals clearly that in the preliminary lesson their opportunity to contribute meaningfully was significantly curtailed.

The summary presented in Table 5-2 above presents a striking contrast between the preliminary and research lessons. In each analytic focus area research lesson activity reflects opportunity for the lower attainer to influence task progress. Data shows however that the difference in the two scenarios arises from relatively subtle difference in the activity of both pupils in the pair. Higher attainers gave more credence to lower attainers’ suggestions; equally lower attainers demonstrated greater assertiveness and confidence in value of the ideas that they articulated.

In the following chapter I discuss the implications of the findings in both analysis chapters and propose a theoretical model for the role of the focus on awareness in shaping the interaction in which pupils engaged.
Chapter 6: Discussion

This research study placed the mathematical learning of lower attaining pupils in mixed attainment pairs at its centre. It was guided by four research questions which are detailed below alongside the main findings arising from each:

**Research question 1:** ‘In what ways does a pedagogical focus on noticing impact on the nature of lower attaining pupils’ mathematical awarenesses?’

a) The focus on noticing was associated with lower attaining pupils frequently demonstrating awareness of structure and generality

b) The focus on noticing had limited impact on lower attaining pupils’ awareness of the mathematical patterns and properties of their findings and results.

**Research question 2:** ‘How do lower attainers’ awarenesses contribute to task progress for the pupil pair?’

c) Lower attainers’ expressed awarenesses of structure and generality made significant contribution to task progress.

**Research question 3:** ‘In what ways does a pedagogical focus on noticing influence the nature of the interaction between lower attaining pupils and their higher attaining partners?’

d) A focus on noticing was associated with more fluid authority relations between the focus lower attaining pupils and their partners

**Research question 4:** ‘In what ways does a pedagogical focus on noticing influence the nature of the expectations for engagement and activity conveyed through class teachers’ introductions and questions?’

e) Class teachers’ introductions and questions in the preliminary lesson reflected a focus on following the rules of the task and ensuring accuracy

f) Class teachers’ introductions and questions in the research lessons reflect authority distributed to pupils to make decisions and establish understanding jointly
In this chapter I present discussion arising from the analysis of the findings presented in the previous chapters. In so doing it is important to recognise the small scale and scope of the study both in the number of focus pupils and in the relatively high mathematical confidence of their class teachers. Whilst not reducing the strength of the findings, the recommendations and conclusions arising from the discussion below require further exploration in a wider range of primary settings.

I begin with research questions 1 and 2 and, building on the positive impact of the intervention on lower attainers’ awareness of structure and generality, I argue for the potential of lower attainers to make valuable contributions to mixed pair working. I also revisit the contrast in findings in the impact of the intervention on awarenesses of different types and identify improvements to the focus on noticing.

Research questions 3 and 4 sought to provide understanding of the way in which the pedagogical focus on noticing influenced the nature of interaction between pupils. The disparity in the impact of awarenesses on task progress reported in chapter 4 between the preliminary and research lessons requires an explanation. Here I present a theoretical proposition that the focus on noticing acts as a mediating influence in the operation of authority relations in the mathematics classroom, enabling the mathematical achievements identified in the findings.

6.1 Awareness and the contribution of lower attainers to mixed pair progress

The facility of mathematically lower attaining pupils to think and operate mathematically was discussed in chapter 2. Here I noted that lower attainers had been clearly shown to be able to generate and extend mathematical ideas, make conjectures, articulate generality and justification and to explore possibilities. Findings presented in 4.1 serve to confirm earlier research findings: in the current study lower attaining pupils repeatedly demonstrated that they were engaged in thinking and operating mathematically. Their actions and speech permitted inference of significant mathematical awarenesses relating particularly to mathematical structure and generality but also on occasion to mathematical processes and mathematical properties. Moreover, these awarenesses were directly
employed by the focus lower attainers in articulating mathematical reasoning, for example through the refuting of conjectures, and in articulating generality. The mathematical behaviours of the lower attainers in this study were thus very much in line with those observed in previous research (Watson, 2005; 2001b; 2001a; Watson and De Geest, 2005; Houssart, 2004; 2002); this study adds further to evidence of lower attaining pupils’ capacity to think and operate mathematically.

Of note in this study is the evidence of lower attaining pupils’ capacity to develop and articulate awareness of structure and generality. In this study, important structural awarenesses, either arising out of specific task constraints as in research lesson 1 ‘Magic V’ or arising from mathematical structural relationships (as in research lesson 3) were first evident in the lower attainers of the mixed pair in three of the four pairs. This is particularly significant in the light of Mason et al’s (2005) view that generality is at the heart of mathematical activity. Lower attainers in this study were, according to this viewpoint, engaging in the most significant aspect of mathematical functioning.

Mulligan and Mitchelmore (2009) identified a strong correlation between awareness of structure and mathematical achievement. This would point towards a likelihood of the higher attainers expressing such awarenesses more readily than their lower attaining partners. My findings, however, reveal lower attainers demonstrating significant structural awareness in advance of their higher attaining partners appearing to run counter to this earlier work. However, in related research (Mulligan, 2011; Mulligan et al., 2006), the authors recommend the active promotion of awareness of structure for low achievers, suggesting that an increased awareness of structure would impact on pupils’ later mathematical development in a range of areas. It is worth noting that the intervention reported by Mulligan et al (2006) emphasised both the use of enactive and pictorial representations and the articulation of awareness. Thus whilst different in many aspects of its structure including the nature of the tasks employed, Mulligan et al’s (2006) intervention shares an underpinning similarity in its pedagogical emphasis to the current study.
The focus of this study on lower attaining pupils working in mixed attainment pairings in primary mathematics classrooms enables a direct and original contribution in relation to the role that lower attainers can play in mathematical learning and progress in such mixed partnerships. In chapter 2, I noted that where research focuses specifically on attainment differences, findings have shown that the use of mixed groups or pairings results in enhanced learning for the lower attainer through their access to a wider range of topics and to the higher level interaction of their peers. These studies thus position learning gains for the lower attainer as arising by virtue of what the higher attainer can offer in the partnership. In the same studies, where higher attainers also gain it is through the requirement for them to articulate their own understanding in response to questions from their lower attaining partners. Whilst undeniably valuable in and of itself, such gains for the higher attainer are seen as arising from their own activity of articulation.

Indeed, the importance of exposure of the lower attainer to the high level interaction that such partnerships can afford should not be discounted. Examination of the interaction between the pupil pairs in the current study reveals occasions in which the higher attainer’s contribution can be seen to support the lower attainer in developing mathematical awarenesses through direct explanation or scaffolding (Wood et al., 1976) of ideas. An example of this between higher attainer Hannah and lower attainer Rose was discussed in 5.2.6. Here the scaffolding interaction reflected the higher attainer’s more secure grasp of the connection between multiplication and division and was used by the higher attainer to explain the thinking required to engage in the task.

Recently published evidence from pupil interviews (Tereshchenko et al., 2018), suggests that in mixed classes lower attainers consider that they have the potential to contribute as well as receive help in their mathematics lessons. Nevertheless, the specific contribution that lower attainers can make to mixed pair working through their ideas, conjectures, or generality is not well evidenced or articulated in existing literature (Taylor et al., 2017). In the current study however, data reveals how mathematical awareness demonstrated by lower attainers directly contributed to task progress for both pupils in the pair. Indeed on several occasions (e.g.4.1.3,
Extract 4-10 and 4-12), lower attainers were able use their awarenesses to respond to their partner’s uncertainty or to an incorrect conjecture that their partner made. Particularly in these instances, lower attainers were successfully engaging in cognitively challenging processes. Firstly, they needed to be able to bring their awareness to the focus of that of their higher attaining partner. A greater willingness of lower attainers to make such attempts has been noted (Sfard and Kieran, 2001). Regardless of this, the process requires an intentional shift of attention (Mason, 2011) from their own focus to that of their partner. Secondly, they needed to listen to, understand and interpret the ideas that their higher attaining partners articulated. Lastly, in articulating a constructive (and mathematically accurate) response that either added to or raised a challenge to their partner’s ideas they needed to engage in the highest level of engagement identified by Webb et al (2014). In some cases (e.g. Rose in 4.1.3), this drew on and extended the language structure that the higher attaining partner had used further evidencing the lower attainer’s understanding of what the higher attainer had communicated and their ability to frame their response in a manner that would be understood by their partner (Webb et al., 2014).

Houssart (2004) notes that the term ‘lower attainers’ presumes that learners have been given opportunities to attain. In this study, the facility of lower attainers to develop and articulate structural awareness suggests that these pupils can engage in the central bedrock of mathematical activity: generality (Mason et al., 2005), when given the opportunity and support to do so. In each of the examples presented in 4.1, material progress in the task was made possible through a process in which the lower attainers played an integral part. The contributions made by lower attainers occurred at a range of points during the lessons and addressed a range of mathematical awarenesses. The range of awarenesses is consistent with the types of ‘whispers’ that Houssart (2004; 2001) noted, but unlike the pupils in Houssart’s study, the communicative intent of the pupils in the current study was unambiguous. Across all focus lower attainers and all research lessons, articulated awarenesses were intended to be heard by their partners and were intended to influence activity. This
reflects a greater confidence in the value of their contributions than others have noted from lower attainers (Black and Varley, 2008; Cobb et al., 2009; Bishop, 2012).

These findings lead me to argue that in mixed attainment partnerships, gains for the lower attainer do not arise purely through exposure to higher level concepts or to higher quality interaction. Evidence from this study is that lower attainers are not simply following along, or internalising the ideas of their higher attaining partners. Whilst benefits to the lower attainer may arise from access to the thinking of a higher attainer, lower attainers in such partnerships can also gain because of the nature of their own mathematical awarenesses and from the reasoning that they build from these awarenesses. Higher attainers similarly can gain from exposure to and use of the awarenesses and reasoning that their lower attaining partners contribute to the task. This represents an original contribution made by this study.

However, the conditions within which such contributions by lower attainers can be made are important to consider and the findings of this study highlight two areas of caution. Firstly, in the absence of a focus on awareness, opportunities for lower attainers’ ideas to impact productively on task progress were limited. The following section considers the events observed in the preliminary and research lessons from a critical realist standpoint and discusses the relationship between the focus on awareness and the operation of the mechanism of authority relations.

Secondly, data presented in 4.2 reveals the limited impact of the intervention on pupils’ awareness of pattern and property. Exploration of this limited impact (4.2.4), highlighted the typically symbolic nature of recording, teacher questioning that did not productively focus pupils’ attention, and pupils’ limited appreciation of the value of awareness of property, as significant in producing this outcome. This finding is particularly useful in making recommendations for practice since each feature represents an opportunity to augment the pedagogical focus on noticing to more productively provoke awareness.
6.2 Authority relations and the focus on awareness

Critical realism proposes that the generative powers of social structures underpin the events and experiences of the social world (Collier, 1994; Bhaskar, 1989; Corson, 1999). In this study, events are teachers’ and pupils’ actions and interactions observed in the classroom; experiences include the awarenesses that have been inferred from these same actions and interactions. In 3.7 I identified the retroductive inference embedded in critical realist analysis which seeks to answer the question ‘what must be the case for phenomenon X to occur and to occur as it does?’ (Sayer, 2010; Danermark, 2002; Lawson, 1998). In this section I propose an explanation of the difference observed in events between the preliminary and research lessons.

Data presented in the two previous chapters demonstrated that the pedagogical intervention was associated with lower attaining pupils’ mathematical awarenesses of structure and generality having greater impact on task progress in the research lessons. It also illustrated other features that were associated with this finding. Firstly, interaction between the higher and lower attainer differed in the research lessons to that which had been observed in the preliminary lesson. Secondly, features of the pedagogical intervention, particularly access to the use of representations, appeared to be significant in influencing the way in which pupils interacted with the task and with each other. Thirdly I also noted that the distribution of authority evident in teachers’ introductions and in their interventions at pupils’ tables differed in key ways between research and preliminary lesson.

In this section I propose a model of the influences operating that connects these features of distribution of authority, access to representations and nature of interaction and explains how collectively these features enabled the mathematical awarenesses of the lower attainers to contribute so productively to task progress in the research lessons.
6.2.1 Distribution of authority through setting of task goals

The distribution of authority in the classroom is reflected in how the teacher positions the pupils in terms of the way that they are ‘entitled, expected and obligated’ (Gresalfi and Cobb, 2006, p51) to operate and to interact as they engage in their mathematical tasks. Situations in which pupils are expected to balance the exercising of their own individual agency in contributing insights, conjectures and decisions about ways of working and recording on the one hand, with drawing on the authority of the discipline and its established methods and practices on the other hand, stand in contrast to situations in which pupils rely on the teacher, or on the discipline of mathematics as sole authority.

Across all lessons the tasks selected were investigative, with multiple solutions and opportunities for differing approaches to be taken. Such tasks offer the greatest affordances for the achievement of individual agency (Boaler, 2016; Boaler and Staples, 2008) and thus in all instances, opportunities to contribute ideas and insights, and to make decisions about working processes existed. However, drawing on Boaler’s appeal for mathematics education to develop ‘a greater understanding of its nuances’ (2003, p3) differences in the ways in which tasks were presented can be seen to have altered the way in which these affordances were realised. Differences in expressed and perceived task goals and in the expectations, entitlements and obligations placed on pupils (Gresalfi and Cobb, 2006) arising from these goals, between the preliminary and research lessons, can be seen to give rise to differing distributions of authority with resultant impact on the nature of pupil interaction.

In the preliminary lesson, pupils were presented with the rules of the task in relation to how the dice needed to be arranged, the criteria for a successful solution and were instructed to record their calculations and solutions (see 5.1.1). The goal was to find the solutions that the task contained; pupil activity was focused on finding these solutions and recording them. This focus on rules and requirements effectively positioned the teacher and the rules of the task as the mathematical authority (Grootenboer and Jorgensen, 2009; Boaler, 2003). Pupils were obliged to check that
their solutions were accurate and that they were in line with the rules. Success was framed in the number of correct solutions found (Boaler, 2002).

The research lessons reflected a different distribution of authority arising from the focus on noticing. This can be seen in the entitlement, expectations and obligations noted by Gresalfi and Cobb (2006) as an indicator of how authority is distributed. The expectation that pupils would focus on what they noticed, together with the obligation to share these noticings and to use them to explain their findings, embedded an entitlement to contribute noticings and ideas in the expectation that they would be valued by their partners and by their teachers.

Voutsina and Ismail (2011) report that their focus on awareness of additive relations altered the goal of the task from solving to explaining for the pupils concerned, increasing their sensitivity to these relations. In the current study this same shift in the goal of the task was observed. Success was framed in what was noticed and how this was used to explain and predict. As such, the authority residing in the rules of the task was balanced with the positioning of all pupils as having the authority to exercise individual and collective agency. This reflects the ‘dance of agency’ (Boaler, 2003; Pickering, 1995), in which individual action reflects a balance between the use of individual ideas and insights and the use of disciplinary procedures. Boaler (2003), drawing on Pickering (1995), identifies that in such situations pupils view their role as to learn about relationships, to generate ideas, questions and justifications. Thus in the research lesson the establishment of a more distributed authority (Gresalfi and Cobb, 2006) influenced the way in which pupils engaged with the task.

Thus I see the focus on noticing as stimulating a particular distribution of authority through the expectations, obligations and entitlements that it embeds. In the model overleaf (figure 6-1), arrows connect different elements indicating the direction of influence of one component on another. Distribution of authority influences, directly or indirectly, all aspects beneath it. In particular, I contend that the focus on noticing acted as a mediating influence in the functioning of authority relations between the pupils in each pair, impacting on both the nature of interaction and on the way that representations were used. This is further discussed in subsequent subsections.
6.2.2 The nature of pair interaction

The way in which authority and influence was constructed through interaction was a particularly notable feature of the study. It shed light on how teachers’ framing of pupils’ obligations and entitlements impacted via pupils’ interaction on the opportunities for mathematical engagement afforded to lower attainers.

Characteristic of the preliminary lesson was a mutually accepted positioning of the higher attainer with intellectual authority (Langer-Osuna, 2016). Across three of the four focus pairs, this manifested in features of unproductive authority relations, akin to those established in previous research. Thus for example, higher attainers intervened to evaluate the solutions or to check accuracy of partial solutions of the lower attainer (Esmonde, 2009b), they provided responses to lower attainers’ questions and ideas that were frequently dismissive or minimal (Wood and Kaliniec, 2012; Esmonde, 2009b) or ignored lower attainers’ contributions completely. Additionally lower attainers were frequently not afforded spatial privilege (Engle et al., 2014); they were not physically attended to when speaking and eye contact was minimal. Together these interactions served to diminish the status of the lower
attainer in the task. Notable across all pairs was the lower attainers’ persistence in attempting to communicate and to offer ideas despite the limited responses of their partners (Sfard and Kieran, 2001). For the higher attainers, the preoccupation in finding and recording solutions and in persisting in so doing, was at the expense of the use of the ideas of the lower attainer. This occurred despite the merit of these ideas and of their potential to establish possibilities and constraints that might make solution finding more efficient and productive.

The three components noted above of ‘entitled, expected and obligated’ (Gresalfi and Cobb, 2006, p51) are significant in understanding how the distribution of authority articulated by the teacher in the lesson introduction, influenced subsequent pupil interaction. In the preliminary lesson, no expectation or entitlement to exercise agency in noticing features, explaining results, raising conjectures was articulated nor was any obligation to discuss ideas with partners. Whilst this did not preclude pupils in engaging in these behaviours, it appears that the teacher’s expressed interest in the number and accuracy of solutions, predominated. This focus on finding solutions led to a mutual acceptance that the higher attainer should assume authority in the search for these solutions with the result that contributions by the lower attainer were frequently dismissed or disregarded and interaction rendered minimal.

The research lessons provide a striking contrast, being characterised by lower attainers demonstrating confidence and a quiet insistence when expressing an awareness or idea, or when seeking to maintain involvement at a particular point. For their part, higher attainers demonstrated a willingness to construct themselves as fallible and uncertain; a feature not present in the preliminary lesson. In addition, higher attainers more readily responded to the ideas of their partner and engaged with their partners in exploring possibilities. The expectation that pupils would discuss, together with the entitlement to do so, is reflected in their dialogue across all classes and all research lessons. However, it is not just that interaction was of a more exploratory nature, it was that it was more equitable. The characteristics above resonate strongly with features of equitable relations identified in earlier research, such as the presence of dynamic sharing of authority and mutual challenge.
(DeJarnette and Gonzalez, 2015) and reciprocal help seeking (Boaler, 2016; Boaler and Staples, 2008; Cohen and Latan, 1995).

Gerson and Bateman (2010) argued that teacher pedagogy should seek means by which the traditional acceptors of authority relations can be empowered. In the context of this study these traditional acceptors are the lower attainers in a mixed attainment partnership; evidence from the preliminary lesson confirms the operation of this authority relation. I argue that the evidence from the research lessons demonstrates that for lower attainers in primary mathematics, a pedagogical focus on noticing is one way in which Gerson and Bateman’s (2010) recommendation can be achieved. In this study, the accessibility of engagement based on awareness was combined with the high profile afforded to this awareness in the framing of the task goals. This meant that not only could the lower attainers meaningfully contribute mathematically, but that these contributions were valued by their partners because of the high status that teachers had afforded them.

In earlier research Cohen and Latan (1999; 1997) argued that teachers should publically assign competence to the intellectual contributions of lower attainers (or other low status students), in order to raise their profile and status in the task. With traditional authority relations prevailing in the preliminary lesson and in the absence of a specific focus on noticing this may have been a useful approach. However, data from the research lessons indicates that lower attainers’ contributions were valued by their higher attaining partners on their mathematical merit, without recourse to this external validation on the part of the teacher. Their evident relevance to progress in the task was sufficient to raise the profile of the lower attainers in this context.

I contend that the focus on noticing and awareness accounts for this difference in the nature of pupil interaction through the difference in expectations, entitlements and obligations (Gresalfi and Cobb, 2006) that it embedded. Through this differing distribution of authority, the focus on noticing effectively democratised engagement and interaction in the task.
The apparent counter example offered by research lesson 2 is important to consider in this discussion. In this lesson, whilst the focus on noticing was emphasised by all class teachers and the task structure was investigative, offering multiple avenues of exploration, early features of the task drew on pupils’ understanding of multiplication and division, specifically, their knowledge of multiple/factor pairs. Here the traditional advantage noted by Boaler and Staples (2008) re-emerged and higher attainers led this aspect of the task. However, the alteration in the operation of authority relations noted in other research lessons persisted here despite this. This task did not result in higher attainers interacting in the dismissive and uncommunicative way that had been observed in the preliminary lesson. Instead, the support provided by the higher attainer was focused on explaining how and why the relationships existed. This was undertaken with a clear intent to empower their lower attaining partners to engage in the task. Rather than being a counter example, this example adds further to the contention that a focus on noticing acts as a mediating influence in the operation of the causal mechanism of authority relations. The focus on noticing ensured that even though a calculation based advantage existed, it was not utilised by the higher attainer to gain control of the task.

6.2.3 Access to and use of representations

In the preliminary lesson, lower attainers were afforded intermittent visual and physical access to the resources of the task; on many occasions higher attainers intervened in lower attainers’ activity to remove task resources ostensibly to check accuracy and then maintained control of them. In many cases their activity was not attended to by their partners. In contrast, the research lessons reflected greater equity in access to and use of representations; visual and physical contact was maintained and with pupils tending to orient themselves towards each other and towards the activity underway, lower attainers’ activity was attended to more closely.

Again the influence operating to create this difference can be traced to the teacher’s framing of the task. Access to and use of representations is also reciprocally
influenced (indicated by the double headed arrow in figure 6-1 above) by the nature of interaction between pupils.

Teacher influence on the use of representations arose from the stated goal of the task. Thus in the preliminary lesson the dice were manipulated in order to find solutions; correct positioning of the dice was part of how teachers evaluated pupil solutions. Teachers did not draw pupils’ attention to the structure of the dice, for example the positioning of numbers on opposite faces or to possibilities and impossibilities arising from this. Conversely, in the research lessons, teacher emphasis on the development of mathematical awareness was associated with the frequent recommendation that pupils used resources to support the development of these awareness. Resources and representations were specifically selected by the teacher on the grounds of their affordances in modelling the mathematical ideas under consideration (Delaney, 2001). Examples of the use of representation stimulated by teacher recommendation have been analysed in the preceding chapters (e.g. sections 4.1.3 and 4.2.1).

Access to and use of representations also influenced and was influenced by the nature of pupil interaction. In the preliminary lesson, pupil use of the dice was in line with their teachers’ instructions. In this lesson the manner in which interaction restricted use of the resource is exemplified in the observation that awarenesses about features of the dice from both Mia and Joe were disregarded. Thus an opportunity to act on these awarenesses and to use the dice to explore possibilities was missed. The frequent interventions in lower attainers’ exploration and higher attainers’ tendency to control access to the dice also restricted the opportunities for interaction in this lesson. Higher attainers’ greater reluctance to communicate their thinking and to operate in their ‘private channel’ (Sfard and Kieran, 2001) whilst exploring possibilities restricted both the quantity and nature of interaction. Thus it seems that the exercising of intellectual authority together with the control of the physical resources that had been assumed and accepted by both pupils in the pairs (Gerson and Bateman, 2010), served to disadvantage both pupils, albeit in different ways.
The use of representations to develop and demonstrate awareness in the research lessons had a strong impact on the interaction between pupils. This arises from classteachers’ consistent emphasis on noticing and their encouragement of the pupils to use the available resources to provoke awareness. In these lessons, representations were used to develop awareness of structure in particular and thus representation use afforded pupils the opportunity to articulate these structures and to use them in constructing reasoning. The finding that the use of representations particularly supported lower attainers (Foote and Lambert, 2011) to develop awareness of and to articulate structure was, I argue, particularly influential in ensuring that interaction in the research lessons reflected a greater equity between the pupils.

6.2.4 Impact of lower attainers’ mathematical awarenesses

Discussion above has asserted that the nature of interaction and access to and use of representations were both influenced by the framing of the task goals and the distribution of authority arising from the focus on noticing and the expectations, entitlements and obligations it embedded. Here I argue that both the nature of the interaction and lower attainers’ access to representations together influenced their capacity to develop and express awarenesses and to enable these awarenesses to impact on task progress. Thus I argue that the focus on noticing impacted on lower attainers because of its mediating impact on the nature of interaction and access to representation.

Data presented in chapter 4 supports this assertion. Here, the evidence of the research lessons is of lower attainers repeatedly developing and demonstrating awarenesses that contributed to task progress. Where interaction with their partner and access to representations was more limited, as was the case in the preliminary lesson, awarenesses were demonstrated but remained undeveloped. The awareness pathway diagrams presented in chapter 4 illustrate this contrast (section 4.1.5).

Hewitt (2009b; 1997) argues that individual lesson experiences that prioritise pupils’ ability to notice and observe render mathematical content accessible to all. This, he argues, arises because engagement does not rely on previous knowledge or stages
learned, but instead builds on children’s natural capacities to become aware of features such as similarity and difference (Gattegno, 1988). Data from the current study which repeatedly shows lower attainers contributing awarenesses gained through their engagement in the task supports Hewitt’s (2009b; 1997) argument. In short, the focus on noticing provided a task goal to which lower attainers had equal access through their capacity to contribute awarenesses. Indeed these findings both bear out and go some way to explaining Hohensee’s (2016) finding that a focus on noticing is particularly beneficial for lower attainers.

I return to the model presented in figure 6-1 to illustrate how the different components combine to produce the difference in the nature and use of awarenesses that was observed. Two annotated versions of this model are shown overleaf: Figure 6-2 represents the situation prevailing in the preliminary lessons; Figure 6-3 represents the evidence of the research lessons.
Task goal: find solutions

Pair interaction reflects authority of higher attainer in managing task in order to find solutions

Limited and interrupted visual and physical access to representation for lower attainers

Awarenesses rarely articulated by higher attainers. Lower attainers’ awarenesses not productively utilised

---

Research lessons: model of influences

Task goal: to notice

Equitable pair interaction focused on developing and using awarenesses. Lower attainers

Access to and use of representations shared; activity of lower attainer attended to

Lower attainers’ awarenesses valued and used productively by both children in searching for and explaining task outcomes

---

Figure 0-2 Influences operating in the preliminary lesson

Figure 0-3 Influences operating in the research lessons
In figures 6-2 and 6-3, each component of the model combines to produce the outcomes in terms of awarenesses observed. The contrasting outcomes for lower attainers illustrated in these two versions of the model reflect the impact of the focus on noticing. Thus, it can be concluded that the focus on noticing, embodied particularly in the teachers’ introduction, and in their encouragement that the pupils should use representations, has had a strong impact on the productivity of awarenesses for the lower attaining pupils through mediating the interaction that took place and the way in which access to representations was managed. Analysis in chapter 4 demonstrated the mathematical strength of lower attaining pupils’ awarenesses and the material contribution of these awarenesses to task progress. In short data shows that their capacity to contribute is not in doubt; thus the focus on noticing has provided a means by which their capacity to do so could be realised.

6.2.5 The nature and affordances of teacher questions

The nature of the authority relations with mathematics that the teacher establishes will influence the nature of interaction in which teachers engage with their pupils (Boaler, 2002). Whilst this finding is borne out in the difference observed between questions asked by class teachers in the preliminary lesson and in the research lessons, this feature is not included in the model above (figure 6-1) since it was not a focus for specific analysis. Teachers’ interaction with pupils was examined in so far as it related to the identified critical incidents and does not reflect all the teacher-pupil interaction that took place. The following thus represents a hypothesis requiring both further research and further analysis of the current data set. Therefore, here I present indications arising from examination of critical sequences and from consideration of sections drawn in to support existing analysis.

In the preliminary lesson, questions posed at pupils’ tables tended to focus on the number and accuracy of solutions, the way that totals had been ascertained and pupils’ confidence that they had not missed any dice faces out in arriving at their solutions. Such questions served to emphasise the obligations on pupils to find and check solutions. The lack of impact of lower attainers’ awarenesses of structure and property may in part be due to teachers’ evident interest in accuracy.
In the research lessons, questioning repeatedly focused on what had been noticed and what such noticings meant. Thus in the research lessons, teachers’ lines of questioning frequently re-emphasised the high value that had been placed on noticing and on explaining what had been noticed. For example, in 4.2.1 the teacher’s continued emphasis on explaining why it was impossible to have 2 at the bottom of a successful ‘Magic V’ required the two pupils to shift their attention back to the properties of the numbers involved and to look for a feature that might explain this finding. Here, the teacher was only minimally interested in the finding that 2 was impossible; her questioning was focused on reaffirming the obligation that arose from the distribution of authority (Gresalfi and Cobb, 2006; Boaler, 2002) and entailed that the pupils should seek to explain this outcome.

At times teacher questioning was a clear attempt to ‘force’ awareness: to steer pupils towards a particular awareness (e.g. 4.2.4, Extract 4-30). At other times the provocation was in the form of a suggestion of where attention may be focused (e.g. 4.2.4, Extract 4-32). The success of such lines of questioning in provoking the desired awarenesses varied; I have discussed, in 4.2.4 and revisited in 6.1 above, that teacher questioning did not always succeed in focusing pupil attention in the most productive manner. Where questioning was more successful, where teacher questions framed the nature of what pupils might attend to, these questions represented stronger affordances than those that invited noticing but without specifying the precise nature of it, in line with earlier research (Gresalfi et al., 2012).

Other findings relating to questioning are also in line with those established in previous research. Thus, data firstly confirms that beyond the initial provocation, teachers rarely participated in the subsequent interaction (Towers and Martin, 2014). Secondly, also in line with Towers and Martin, (2014) data shows that such questions were also offered at moments where it appeared that something that was available to notice maybe had not been, or when the pair appeared to have stalled in their thinking.

Consequently, I hypothesise that teacher questioning acts as a further influence on both the development of awareness and on the nature of interaction. Indeed, on
several occasions, data shows that an awareness was demonstrated following a teacher’s provocation. It cannot be demonstrated that without such provocation an awareness would not be forthcoming, indeed the preliminary lesson shows that this can be the case. However, since pupils listened and attempted to respond to their teacher’s suggestions and questions, even when the desired focus of attention was unclear, teacher provocation did appear to influence the focus of pupil attention at particular times and thus it is likely that such questioning provided a spring board for awarenesses to emerge.

The following chapter draws together the outcomes of this discussion chapter, identifying contributions to theoretical understanding and making recommendations for primary practice.
Chapter 7: Conclusions and recommendations

This research was guided by two aims. These were:

1. To contribute practical and theoretical understanding of the role of awareness in primary lower attainers’ mathematics learning and make recommendations for primary practice
2. To contribute to understanding of the relationship between pupil interaction in a mixed attainment pair and opportunities for the lower attainer to contribute mathematically

In this chapter I first detail the contributions and recommendations arising from this research study in relation to the aims above and to the methodological approach of the study. Subsequently, I consider the limitations of the current study and then suggest further research building on the outcomes and understanding gained thus far. I conclude this thesis by reflecting on my own learning throughout my doctoral study.

7.1 Contributions to knowledge and understanding

In 2.1.4 I noted that current research evidence relating to the impact of mixed attainment groups is limited (Taylor et al., 2017). Evidence available identifies the potential benefits for, but not the contribution made by, lower attainers in these arrangements. The outcomes of this research constitute important evidence of the contribution that lower attainers can make to mathematical progress for themselves and their higher attaining partners. As such it represents an original contribution made by this study.

A second contribution derives from clear impact of the pedagogical focus on noticing in enabling the mathematical awarenesses of lower attaining pupils to have impact on task progress. Whilst authors have both promoted the value of a focus on awareness (Hewitt, 2009b) and hypothesised its benefits for lower attainers in particular (Hohensee, 2016), this study contributes classroom based evidence of the impact of this focus on progress for mixed attainment pairs. It also contributes an analytic framework devised to support inference of awareness through identifying a set of behaviours that support inference of pupil awareness of mathematical ideas, relationships and processes.
Thirdly, arising from the mechanism by which the pedagogical focus achieves its impact, the study presents a methodological contribution. Analysis of the nature of pupil action and interaction drew on features of the influence framework devised by Engle et al (2014) and on the typology of authority relations proposed by Gerson and Bateman (2010). The resultant analytic framework led to the development of a model of influence that includes the teacher’s framing of the goals of the task and positions these as instrumental in shaping the ensuing pupil action and interaction. As noted in 6.2.5, this model requires further research to refine its components and to establish the role of teacher questioning in shaping pupil activity. However, it represents a contribution in its potential to provide a framework with which to interpret influences operating in collaborative mathematical activity in order to support understanding of pupil outcomes.

7.2 Recommendations

The outcomes of the study enable this thesis to support the recommendation made by a range of authors (Anthony and Hunter, 2017; 2016; Marks, 2016; 2014; Boaler, 2010; 2008) for a move away from groupings based on notions of ‘ability’ and towards the use of groups and pairings that do not pre-judge a pupil’s potential to make valid and meaningful mathematical contributions to the tasks in which they are engaged.

However, this recommendation has an important caveat. The second contribution made by this study is the evidence that a focus on noticing and awareness, made explicit through the expectations, entitlements and obligations (Gresalfi and Cobb, 2006) necessarily placed on pupils, is an avenue by which the potential of mixed attainment groupings can be productive for both pupils. Outcomes from this study indicate that without such overt emphasis on the importance of noticing from the class teacher, mixed attainment pair activity can succumb to traditional authority relations that serve to disadvantage the lower attainer and which limit progress for both pupils. Thus, a second recommendation made by this thesis is that pupils’ mathematical awareness is afforded a high priority in primary mathematics
pedagogy within the context of mathematics learning in mixed attainment pairs or groups.

Indeed, whilst the focus on awareness in this study has been in the context of the use of open structured tasks, the recommendation for a focus on awareness transcends all task types. In short, if there is nothing in a mathematical task that it is useful for a pupil to notice, then the value of the task must be in question. If there is something to notice, then it is important that pupils’ attention is directed towards it.

A third recommendation arises from the limited impact of the focus on noticing on pupils’ awareness of mathematical patterns and properties. I propose that greater attention is given to provoking awarenesses of this kind through the use of enactive representation and carefully structured teacher questioning.

In making the recommendation for a focus on noticing as a means to enable lower attainers to contribute meaningfully in mixed attainment pairs, I am mindful of the policy context in which primary classroom teachers operate. In the time since this classroom data was collected, the UK government’s recommendation and support for a ‘Mastery approach’ (NCETM, 2014) to teaching mathematics in England’s primary schools has gathered momentum. My recommendation of an increased focus on pupil awareness is not inconsistent with either the aims or the recommended approaches to teaching for mastery. Indeed, recent guidance focusing on the importance of providing both conceptual and procedural variation consistently emphasises the need for teachers to ask pupils ‘What is the same? What is different? What do you notice?’ in order to support them to become aware of relationships, properties and to develop generality (Jacques, 2015). Whilst many of the tasks used under the auspices of the Mastery approach will not conform to the open structure of those used in this study, the awarenesses that lower attainers have shown in this study are of a similar nature. My study, examining pupil activity and interaction in depth, clearly evidences the capacity of primary lower attainers to engage in such activity.
7.3 Limitations of the research

Firstly, the close analysis employed in this study has necessitated focus on a small number of pupils from just two year groups within KS2. Moreover, the sample of classrooms is drawn from one small geographical region of England and the class teachers were all confident graduates of the MaST programme. Broadening of the sample across a larger number of pupils and classrooms and across a wider age range would support deeper and more nuanced understanding of the areas and relationships discussed and provide greater security in generalising from the findings.

In 2.2.6 I highlighted the internal nature of the construct of awareness and noted the caution presented by authors in inferring awareness from activity and interaction. Throughout analysis I have sought to infer with extreme care and to exercise caution in the inferences made. Similarities in trajectories of awareness between pupils give greater confidence in the inferences made, as does the use of the combined evidence of speech, activity and gesture and the application or re-emergence of indicative behaviours at later points in the data corpus. Further classroom video-based research into pupils’ mathematical awareness with a larger research team would add the opportunity for inter-researcher corroboration of interpretations made.

It was not possible, in the context of school timetables and pupils’ availability, to schedule interviews with pupils to discuss their progress in the lesson and to probe their interpretation of the operation of their mixed attainment pair. Such interviews would have afforded an important element of pupil voice and perspective on the interpretations available to me in analysis. However, it should also be noted that such interviews would rest on pupil memory of events and their own awareness of their awarenesses and are not without their own complications.

My decision to use video recording was essential in capturing the detail of classroom activity and interaction. Indeed the subtle differences revealed in chapter 5 suggest that without this facility many different interpretations may have been drawn, particularly with reference to the nature of pupil interaction. However, the presence of the video recorders has the potential to distract and to alter the events of the
classroom (Lankshear and Knobel, 2004) and as such may have generated unrepresentative sequences. Pupils did, on occasion, make reference to the camera, and occasional waving at it was observed. However, a measure of off task talk was noted, as was the independent decision to alter the task on more than one occasion. Thus the camera appears to have recorded an expected range of pupil behaviours. Significantly, the presence of cameras cannot be responsible for the differences observed between the preliminary and research lessons and thus I consider that the significant advantages of this mode of data collection outweighed its limitations.

Lastly, each of the tasks used sought to provide affordances for noticing and to meet pupils’ learning needs across the three classrooms. Appendix A shows the analysis undertaken to ensure that as far as possible, affordances for tasks met the open structure (Boaler and Staples, 2008; Boaler, 2016) aim detailed in 2.1.4. Classroom experience of these tasks across the different sites and with slightly differing teacher approaches meant that exact correspondence of affordances offered and realised across tasks and classrooms cannot be assured.

**7.4 Further research**

In addition to offering contributions and recommendations, this research raises questions that further research can explore. My finding that the intervention had limited impact on awareness of property and pattern stimulates the need for developments to the intervention utilised in this research and its new trial in classroom contexts. This would form a natural next step of the Education Design Research approach (Plomp and Nieveen, 2013) and constitute a second cycle of the existing research study. This second cycle might seek to answer the research question:

*How can a focus on noticing support the development and expression of lower attaining pupils’ awareness of mathematical pattern and property?*

This second cycle might incorporate sub questions relating to the role of enactive representation and teacher questioning which I have highlighted as areas for further attention in relation to this awareness type (section 4.2.4).
I noted above that my recommendation for a consistent pedagogical focus on noticing is in line with the current UK policy context in primary mathematics education. However, little is known about the way in which approaches purporting to offer conceptual and procedural variation afford productive opportunities for lower attaining pupils to develop and use mathematical awareneses. Thus, a second branch of research into awareness would focus on awareness in the context of teaching for mastery and might draw on the opportunities afforded and realised by primary teachers in their use of published resources such as text books to provoke and build on children’s awarenesses. This research might seek to answer the question:

_In what ways does the policy context of teaching for mastery support a productive focus on mathematical awareness?_

This research contributed to knowledge regarding the potential of lower attaining pupils to contribute to mixed attainment pairs, however the evidence base in relation to the successful operation of mixed pairings and groups remains limited. Further research in this area, building on a continued pedagogical focus on noticing, might take the form of case study and draw on pupil voice, employing video stimulated recall of lessons to prompt discussion of the operation of mixed pairs. It might seek to answer the research question:

_How do pupils experience a focus on mathematical awareness in a mixed attainment pair?_

Lastly, this thesis incorporated diagrammatic representations of pupils’ awareness pathways (chapter 4). These showed a strong degree of similarity across focus pupils within each of the tasks used, and I conjectured that this represented the affordances of the task rather than pupil specific development. However, this conjecture requires further exploration. Further video observation of pupils of different attainment level, engaged in paired activity, would enable this conjecture to be refined and would support greater understanding of the trajectory of mathematical awareness in different task contexts. Questions such as the following could frame this research:
What trajectories does the development of mathematical awareness follow over the course of a mathematical task?

7.5 Personal learning and reflection

My doctoral study has produced learning for me in a range of different areas. Below I briefly detail some of the most significant.

Firstly, in relation to my learning about classroom mathematical activity, I have been surprised and consistently fascinated by the nature of activity and interaction captured through classroom video recording. What this first confirms to me is that my interest in children’s mathematical learning is insatiable and I find this reassuring. It also confirms to me the value of video recording as a research tool in understanding the complexities of classroom mathematical activity. The subtleties of gesture, action and speech cannot be captured through any other data collection approach, yet their significance for the reality of the mathematical opportunity of the pupils involved has been made evident through this study. This leads me to conclude that video-based study is essential to further understanding of classroom activity. I am aware that this view is informed by a critical realist perspective that foregrounds the social nature of classroom learning.

The use of video has presented methodological challenges at every turn, from the practicalities of sourcing suitable equipment, to the physical challenges of setting equipment up safely in a primary classroom, to the intellectual challenge of establishing a means of data analysis. Each stage of this has deepened my own awareness of ethical issues, of what constitutes rigour and of transparency in the research process; my skills and knowledge as a researcher are richer for the experience.

At several points in the last few years I have reflected on the trajectory of my own attention and interests that have led to the identification and development of the research focus for the current study. I can only describe this as a process of circling in rather than of narrowing down. This could be interpreted as indecision, as reflective of a broad range of interests, or as evidence of a short attention span. I
consider that it reflects my understanding of learning as complex; focus on one area necessitates a reduction in focus on another. Whilst seeking the detail and depth required to contribute meaningful knowledge, I am acutely aware of aspects that have to be minimised in order to achieve this focus. Learning to research, in common with all other learning requires a process of stressing and ignoring, so it turns out.

Through this process I have also learned to deal with uncertainty, to sit more comfortably with the knowledge that an awareness about the process is emerging, but just out of reach. I have learned to be patient, to let ideas settle, to look from different angles. Indeed, much of the advice to those experiencing being stuck in a mathematical problem solving process (Mason et al., 2010) has applied during doctoral study.

My chief reflection however is that, when work commitments have permitted me to immerse myself for days at a time, I have relished the opportunity to think, analyse, write and reflect that this immersion has offered. Engagement in doctoral study was a firm intention of mine when I joined the University of Brighton in autumn 2006 and there are many reasons why the start, and indeed the completion of this study period did not fit my original ambition. My enjoyment of this process, despite its many challenges, means that I will not now be deterred from continuing to engage in mathematics education research; with greater experience and knowledge of its processes and demands I feel more confident both to make this assertion and that the research that I conduct can make meaningful contributions to the mathematics education field.
References


Kwon, O. N., Ju, M-K., Kim, R. Y., Park, J. H. and Park, J. S. (2013) "Design research as an inquiry into students' argumentation and justification: Focusing on the design of the intervention" in Plomp, T. and Nieveen, N. (Eds.), Educational design research - Part B: Illustrative cases, Enschede, the Netherlands: SLO.


Marks, R. (2013) "'The blue table means you don't have a clue': the persistence of fixed-ability thinking and practices in primary mathematics in English schools", Forum (for Promoting 3-19 Comprehensive Education), 55 (1) pp. 31-44.


Nrich (2015b) 'What numbers can we make?', [online], Cambridge: University of Cambridge. Available: <https://nrich.maths.org/7405> [Accessed 17.03.18].


School of Education (SoE) (2011) *Research Ethics Tier 1 approval protocol*, University of Brighton: School of Education.


Sfard, A. (2002) "There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning" in Kieran, C., Forman, E. and Sfard, A. (Eds.), *Learning Discourse: Discursive Approaches to Research in Mathematics Education*, Dordrecht: Kluwer.


Watson, A. (2001b) "Low attainers exhibiting higher-order mathematical thinking", Support for Learning, 16 (4) pp. 179-183.


Appendices

Appendix A analysis of task affordances for noticing

Preliminary lesson: ‘The Dice train’

<table>
<thead>
<tr>
<th>Awareness of mathematical processes</th>
<th>Awareness of mathematical pattern and property</th>
<th>Awareness of mathematical structure and generality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Awareness of relevance of processes</strong></td>
<td><strong>Awareness of features of the recorded outcomes of successful trains</strong></td>
<td><strong>Awareness of organisation, relationship between properties</strong></td>
</tr>
<tr>
<td>For example:</td>
<td>Awareness that:</td>
<td>Awareness that:</td>
</tr>
<tr>
<td>• fixing a variable: this could arise through for example:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a) fixing the funnel die as a particular number and manipulating the remaining dice to explore possibilities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b) fixing a number on one or other carriage dice, or</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c) identifying a total for the carriages and exploring ways that this could be achieved. This is similar to a) above</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• building on any of the above, pupils may demonstrate awareness of systematic manipulation of the variable to be fixed, eg increasing it successively by 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• random specialising as an alternative or addition to fixing a variable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• exploring modification to an unsuccessful attempt to improve it rather than starting completely afresh</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• checking that a new solution is unique and meets the criteria</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• establishing an order in which to engage in calculating and recording the spot total; checking accuracy of addition</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• the largest total possible on the funnel die constrains the totals possible on the carriage die (carriage die cannot sum to more than six because the funnel die is a 1-6 die)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• a 1-6 die is organised such that numbers on opposite faces sum to 7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• the positioning of numbers on the faces of the dice is connected with the impossibility of making some arithmetically possible solutions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• the structure of the train means that different numbers of spot faces are visible on different dice.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• there is a relationship between size of spot total and train funnel total (smaller spot total on the 2 total train; largest spot total on the 6 total train)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Research lesson 1: ‘Magic V’

<table>
<thead>
<tr>
<th>Awareness of mathematical processes</th>
<th>Awareness of mathematical pattern and property</th>
<th>Awareness of mathematical structure and generality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Awareness of relevance of working processes</strong></td>
<td><strong>Awareness of features of outcomes of successful V shapes</strong></td>
<td><strong>Awareness of organisation and of relationship between properties</strong></td>
</tr>
<tr>
<td>For example:</td>
<td>For all number sets:</td>
<td><strong>Awareness of:</strong></td>
</tr>
<tr>
<td>• Fixing a variable:</td>
<td>• Only one solution is possible for each base number</td>
<td>• the V as a set of five separate numbers in a V shape (pre-structural)</td>
</tr>
<tr>
<td>a) in the context of this task the most useful approach would be to fix the base number of the V and manipulating the remaining 4 cards in a search for solutions</td>
<td>• For any set of five consecutive numbers, three solutions are possible</td>
<td>• the V as a set of two lines of 3 numbers making a V shape (partial structure)</td>
</tr>
<tr>
<td>b) fixing a number might also take the (less useful) form of fixing a number in a different position on the V and manipulating remaining cards</td>
<td></td>
<td>• the V as consisting of 2 arms (one number pair in each arm) connected by a shared base number (structural)</td>
</tr>
<tr>
<td>• building on the above, pupils may demonstrate awareness of systematic manipulation of the variable to be fixed, eg increasing it successively by 1</td>
<td></td>
<td>• the relationship between odd/even properties of number set and odd/even base number property</td>
</tr>
<tr>
<td>• Random specialising as an alternative or addition to fixing a variable</td>
<td></td>
<td>• the relationship between base number size and arm total (larger the base number, the larger the arm total)</td>
</tr>
<tr>
<td>• Checking whether more than one solution is possible before altering the fixed variable</td>
<td></td>
<td>• the relationship between arm totals and number set (middle arm total is three times the middle card in range of 5); use of this to predict an arm total range for a new number set, or to predict the card range needed to give a selected arm total</td>
</tr>
<tr>
<td>• Checking that each solution is unique and has not swapped numbers within an arm</td>
<td></td>
<td>• the relationship between arm totals and number set (middle arm total is three times the middle card in range of 5); use of this to predict an arm total range for a new number set, or to predict the card range needed to give a selected arm total</td>
</tr>
<tr>
<td>• Checking addition of each arm is accurate</td>
<td></td>
<td>• the relationship between arm totals and number set (middle arm total is three times the middle card in range of 5); use of this to predict an arm total range for a new number set, or to predict the card range needed to give a selected arm total</td>
</tr>
</tbody>
</table>
Research lesson 2: ‘Remainders after division’

<table>
<thead>
<tr>
<th>Awareness of mathematical processes</th>
<th>Awareness of mathematical pattern and property</th>
<th>Awareness of mathematical structure and generality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Awareness of relevance of working processes</strong></td>
<td><strong>Awareness of features of outcomes of card pairing</strong></td>
<td><strong>Awareness of organisation and of relationship between properties</strong></td>
</tr>
<tr>
<td>For example:</td>
<td>For pairings giving remainder of zero, awareness that:</td>
<td>Awareness of/that:</td>
</tr>
<tr>
<td>• Fixing a variable:</td>
<td>• successful pairs are ones where the two numbers are in the same times table</td>
<td>• the multiplicative relationship between the numbers in a pair which divide to give no remainder (the smaller card must be a factor of the larger)</td>
</tr>
<tr>
<td>a) In the context of this task this may take the form of fixing the size of a divisor and identifying one or several possible dividends, or</td>
<td>• some cards can be used in more than one pair</td>
<td>• the relationship between card value and the potential of a card to be used as either a dividend or a divisor (cards with values 11 or more can only act as dividends)</td>
</tr>
<tr>
<td>b) Fixing the size of the dividend and identifying possible divisors</td>
<td>• four cards are always left over</td>
<td>• the greater the number of factor relationships a number has the larger the number of possible pairings it can be part of</td>
</tr>
<tr>
<td>• For either of the above, selection of the smallest, or largest pairing of pair for the fixed variable may be an approach (e.g. select divisor of 2 and look for smallest possible dividend, i.e. 4 or largest dividend i.e.20)</td>
<td>• three of the left over cards are prime numbers</td>
<td>• three of the four prime numbers larger than 10 must appear in set of unpaired cards</td>
</tr>
<tr>
<td>• building on any of the above, pupils may demonstrate awareness of systematic manipulation of the variable to be fixed, e.g. increasing it successively by 1</td>
<td>• not all prime numbers are left over</td>
<td>For pairings giving largest total remainder awareness that:</td>
</tr>
<tr>
<td>• Random specialising: the selection of cards in no particular order, guided possibly by awareness of multiplication facts</td>
<td>• smaller prime numbers (values &lt;10) can be paired</td>
<td>• to get a remainder you need a divisor that is not a factor of the dividend</td>
</tr>
<tr>
<td>• Re-arranging of card pairs to increase the total number of pairs made</td>
<td>• one larger (values&gt;10) can be paired with card 1</td>
<td>• The remainder in any pairing cannot be larger than the size of the divisor</td>
</tr>
</tbody>
</table>

For pairings giving largest remainder, or remainder =1: |
• Successful pairs are ones where cards are not in the same times table |
• Prime numbers can be used in pairings which seek remainders |
• to get the largest possible remainder, dividend should be one more than half of the divisor or 1 less than twice the divisor with
largest possible difference between dividend and divisor

- the constraints of the card set means that the largest remainder possible is 9; this can be achieved in two different ways

For pairings giving a specific size (n) of remainder, awareness that:

- Successful pairs conform to relationship that divisor multiplied by selected factor plus n gives remainder of size n
Research lesson 3: ‘What numbers can we make?’

<table>
<thead>
<tr>
<th>Awareness of mathematical processes</th>
<th>Awareness of mathematical pattern and property</th>
<th>Awareness of mathematical structure and generality</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Awareness of relevance of working processes</strong></td>
<td><strong>Awareness of features of numbers made</strong></td>
<td><strong>Awareness of organisation and of relationship between properties</strong></td>
</tr>
<tr>
<td>For example:</td>
<td>Awareness that:</td>
<td>Awareness that:</td>
</tr>
<tr>
<td>• fixing a variable: in the context of this task this could take the form of selecting a number from one bag and exploring the possible ways to select two further cards</td>
<td>• totals can be both odd and even</td>
<td>• each number in the set can be viewed as a multiple of 3 plus one</td>
</tr>
<tr>
<td>• building on the above, pupils may demonstrate awareness of systematic manipulation of the variable to be fixed, selecting two numbers from the same bag, or selecting a single number from a different bag</td>
<td>• individual totals are in several different times tables</td>
<td>• when three such numbers are summed, the three remainders sum to three and this makes the whole number a multiple of 3</td>
</tr>
<tr>
<td>• random specialising: selection of three numbers with no particular organisation of selection</td>
<td>• all totals are in the three times tables</td>
<td>• altering the number of cards selected from the set will alter the properties of the totals made</td>
</tr>
<tr>
<td>• checking addition of the total of the three cards</td>
<td>• all totals are less than 30</td>
<td>• systematic alteration of the numbers in the card set (e.g increasing each by one) will generate a predictable difference in the outcome achieved</td>
</tr>
<tr>
<td></td>
<td>• the smallest total is 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• not all totals between 3 and 30 can be made</td>
<td></td>
</tr>
</tbody>
</table>
Appendix B Data sample: detailed description

<table>
<thead>
<tr>
<th>Time</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>11:54-12:39</td>
<td><strong>CS2</strong> Mia suggests they change the number at the bottom and tries to put 5 there. Ruby says they should be able to find more Vs with 1 at the bottom. She swaps two cards then sums arms and indicates to Mia that one is 7 and one is 8 (its actually 9). Mia drops head to table momentarily. She begins then to suggest a swap to the number at the bottom; Ruby says 'wait' and makes a different swap recreating a V with arms of 8. She says that they've just done that one and sweeps cards together, groaning. Then Ruby says that they should go on to '2' now.</td>
</tr>
<tr>
<td>12:39-14:11</td>
<td><strong>CS3</strong> Ruby lays out cards with 2 at the bottom and swaps pairs of cards in attempt to find solution. Mia watches intently. Ruby tries several swaps then asks which they haven’t tried changing. After a moment more Mia says she knows something that will work and pulls out the 3, using it to form the base of a new V with 5,1 on one side and 2,4 on the other. She says she knows she is right. They check solution together summing the pair on the arm then adding the 3. Ruby says 'yeah' and gives Mia her pen to record new solution.</td>
</tr>
<tr>
<td>14:11-15:49</td>
<td><strong>CS4</strong> T asks if they have noticed anything that could be recorded on sheet. Ruby says that think its impossible to have one with 2 at the bottom. T asks if they are sure about this or if they just think it to be the case. They say they think it is so. T asks them to record this. R asks what they know is possible. Mia describes the two solutions, listing all numbers used whilst pointing to them. T asks them, given that they have said that 2 is not possible, to say what is possible at the bottom. They say 1 and 3 so far. Ruby records on the noticing sheet that not possible to have 2 at the bottom.</td>
</tr>
<tr>
<td>15:49-17:53</td>
<td><strong>CS4</strong> Ruby suggests that there could be more with 3 at the bottom and tries to manipulate, swapping pairs horizontally. Mia notes that the solution created is not correct referring to the sum of just the pair of numbers in the arm, disregarding the bottom number in this. She says look these are 7 and these are 5, indicating pairs with split first and middle fingers. Mia swaps the next pair horizontally. Through this they create an identical but mirrored solution to the one recorded, realise this then Ruby suggests moving on to 4. Cards swapped, Ruby also now just adding pairs on each side. She says she doesn’t think it is possible to have 4 at base. Mia says 'wait' then draws out the 5, uses it to form base of new V and completes a correct solution. They check, Mia picks up pen then Ruby asks if she can record and Mia passes paper and pen to her. she records solution.</td>
</tr>
<tr>
<td>17:53-19:03</td>
<td>T asks what the totals of the arms of the solutions found so far are. Girls count this up and T asks them to record this. Ruby records them, says 8,9,10, hmm. Then says she bets next one will be 11. T asks them to think about what they notice about the totals and the Vs that they have made. Mia taps the numbers and says look 8 tap tap. Ruby says that they need to see what the next one is.</td>
</tr>
</tbody>
</table>
### Appendix C: Critical sequence transcript

<table>
<thead>
<tr>
<th>CS no. &amp; time</th>
<th>CS context</th>
<th>Dialogue and interaction</th>
<th>Awareness pathway</th>
<th>Awareness and authority notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>CS3 12:39-14:11</td>
<td>1. Mia: ‘2 at the bottom’ Mia watches Ruby, hands on table but not touching cards. Ruby pulls 2 into place, with 1 and 3 adjacent to it on either side</td>
<td>This is the beginning of recognising that the base number does not need to be added into the total in establishing if a solution has been found or not. Ruby sums all three cards in turn 2; Mia the arm pair first in turn 3, picked up by Ruby in turn 4. Mia’s awareness of importance of arm pair is fully demonstrated in turns 9 and 11.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2. Ruby pulls 5 and 4 into place to complete V: ‘I know this don’t equal’; with split fingers indicating the arms she says: ‘7, 8 so...let’s say..’ she swaps cards and re-adds indicating each arm with downwards sweeping gesture 9, um..8 um yes 7 I don’t’</td>
<td>Ruby sums all three numbers, voicing only the total of the 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3. Mia smiling bangs fist on table and in response to Ruby’s uncertainty adds the cards in one arm. She points to the 4 then 1 as she says ‘5’ then taps twice to count on 2: ‘6,7’</td>
<td>Mia: sums arm pair first – more organised addition, hint at awareness that base number adds same to each arm</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4. Ruby begins to switch cards around so Mia stops. Ruby sweeps finger across the pair as she says ‘7’, then taps the table twice as she adds base number (2) and says ‘9’. She repeats sweep for second arm pair ‘6’ then pauses as she taps table twice mentally adding base number. Mia glances towards her, smiling</td>
<td>Ruby follows suit</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5. Mia says ‘Oh no!’ at same time as Ruby says ‘8’, Mia smiles and both lean back then forwards again, Mia making yogic like arm gesture of calm at which Ruby smiles</td>
<td>Mia awareness of arm pair equivalence suggested here again – she knew outcome before Ruby voiced it</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6. Ruby again starts to swap cards. Mia touches and draws the 4 towards her saying: ‘I know you’ve already done it. I know, let’s try’.</td>
<td>Mia: awareness that check has been done; sees a solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Ruby moves to draw the card back and adds it to the 3 it is near, tapping twice to add the 2 as bottom number of the V: ‘7..8,9’ She groans as she taps near the 5 and 1 in recognition that the other arm will not equal the same.</td>
<td>Ruby awareness that adding 2 will not result in equivalent arm pair – stronger suggestion of awareness of arm pair equivalence</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Ruby: ‘Erggh, what else can we do?’ She makes a horizontal card swap: ‘We’ve done that haven’t we? Have we? Yeah’</td>
<td>Ruby process awareness - checking</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Mia: ‘I think something might work, like’ Mia pulls the 3 from the arrangement, drawing it towards her and positions it to be the base number. Ruby watches. Mia: ‘5’. She pulls the 5, then after a pause, the 1 to make one arm, saying ‘6’. She then picks up the 4 and 2, dropping and retrieving a card, ‘I know it will work’, before placing the 4 and 2 to make the second arm ‘yeay! I was right’</td>
<td>Construction of V using two equivalent pairs Suggestion of awareness of structure in approach to solution finding Confirmation CS4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Ruby indicates the arm pair of 4 and 2 with her hand: ‘6’</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Mia sums the right arm pair total saying, with emphasis, ‘6’ then completes addition of 3 by counting on ‘7,8,9’. She then waves her hand above the 5 and 1 of the left arm pair again saying with emphasis ‘6’ then adding on the remaining 3 by tapping the table near the 3, three times as she counted: ‘7,8,9’. She turns and looks to Ruby as she completes this.</td>
<td>Justifying a solution – doesn’t need to add the 3 but is doing so to demonstrate what she knows. arm pair totals are the important equivalence addition of the base number adds the same to each arm</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix D: Project information sheet

Name of researcher: Nancy Barclay

Title of enquiry:

Developing reasoning through a focus on mathematical noticing: how can primary teachers support children to notice and use mathematically relevant ideas?

This research is being undertaken as part of my doctoral study at the University of Brighton. The aim of this enquiry is to gain as full an understanding as possible as to how children, particularly lower attaining children, can be supported to notice mathematically relevant ideas when they are engaged in mathematical activity as part of a mixed attainment pair. On a very basic level this could involve a child focusing on the idea that a shape has right angled corners rather than that it is blue; frequently it would relate to the recognition by a child of particular numerical relationships. This enquiry recognises the importance of reasoning in primary mathematics learning and is founded on the simple premise that ‘what you do not notice, you cannot act upon’ (Mason 2002). This is a small scale project in which I hope to work with three teachers over the course of the academic year 2014-15 and to collect data in all three teachers’ classrooms.

Classroom activity will provide the prime data source and I will want to be able to voice record the whole class dialogue when you work with the class altogether at the beginning of the lesson. When children work at their tables I will want to video record selected focus children as they work with a partner. I am particularly interested in how the mathematical noticing of lower attaining children within these pairs develops. I will also observe throughout the lessons, noting features that help me to interpret the audio and video recordings later on. To conduct the research I anticipate making five visits to your mathematics lessons over the course of the year. These will be on mutually agreed dates and times during October 2014 to June 2015. The first visit is introductory and serves to test equipment and meet the children. Following this a baseline lesson will gather data relating to how children currently work together and then three further research lessons to gather data in response to the enquiry question.

This study involves introducing some particular approaches that aim to support and develop children’s facility to notice and use mathematically relevant ideas in their reasoning. These approaches include the use in the research lessons of particular questions (which I hope will encourage productive noticing) as you work with the whole class, the use of resources and a careful selection of activities where we are sure that there are good opportunities for noticing and reasoning. I will ask you to attend a set of four, one-hour workshop meetings with myself and the two other teachers taking part during the course of the year. During these workshops we will review short sections of video of children working to enable us to reflect on their
noticing and their dialogue and activity, and to discuss how to refine or shape the project as the year continues. I will reimburse any travel expenses to attend these workshop meetings.

**Benefits to you and your school**

I hope that being part of this research project will be of interest and benefit to you. It offers the opportunity to work with a small group of teachers to develop provision for children and to understand their mathematical noticing and reasoning in ways that I hope will be productive for them and for you.

**Consent:**

In a study such as this, informed consent from all parties is essential. I will discuss with you and prepare consent forms for you and any other adults who will be in the classroom, and to send home with all children in the class. The consent forms will outline the purpose and process of the study and will provide assurance of confidentiality and anonymity.

All involved, both teachers and children, have the right to withdraw at any point without disadvantage. Data collected up until this point may still be subject to analysis in relation to the enquiry focus.

**Next steps:**

If you are interested in taking part please email me with details as follows:

- Your school name, year group you will be teaching in September, approx class size.
- Whether mathematics in this year group is taught in set groups or mixed attainment classes and if a set group which set you will teach.

This is an expression of interest at this stage and enables us to discuss the project further. You make no commitment by expressing interest. If you are interested to proceed then at a later stage I will also need the following information:

- Approximate number of children at each NC attainment level. This will provide a guide for me in understanding the spread of attainment in relation to year group expectations.
- Whether you regularly have any other adults supporting mathematics lessons and if they are particularly designated to work with specific children. (If you regularly have other adults supporting learning we will need to ensure that they are happy and willing to take part as their dialogue with children may well be part of the data that is collected.)

I will also arrange to meet with your head teacher to discuss and hopefully agree the project; at this stage it may help if your HT is aware of your interest in being involved.

**Selection of research classes**
If, following discussion, more than three teachers confirm interest then selection will be based on the attainment distribution of the class and the geographical location of volunteering schools such that travel to the planned workshop meetings is manageable for all. All volunteering teachers will be contacted once this is complete.

*If you have any questions that you wish to raise, please contact Nancy Barclay on Brighton (01273) 643404 or at n.barclay@brighton.ac.uk.*

*Research supervisors:*

*Dr. Nadia Edmond: n.edmond@brighton.ac.uk; Dr Els De Geest: enfdegeest@gmail.com*
Appendix E: Teacher consent form

Developing reasoning through a focus on mathematical noticing: how can primary teachers support children to notice and use mathematically relevant ideas?

♦ I agree to take part in this research which is to explore how primary teachers can support children’s mathematical noticing

♦ I have read the information sheet and I understand the procedures and the time commitment involved.

♦ I understand that Nancy Barclay will observe, audio and video record a series of mathematics lessons in my classroom

♦ I agree to attend four workshops with the two other participating teachers and Nancy Barclay where we will discuss the progress of the project

♦ I understand that the audio and video recording will be kept securely by Nancy Barclay and destroyed once the enquiry is completed. The audio recording will not be accessed by any other individual. Sections of video will be used to support workshop meetings which only the three participating teachers will attend

♦ I understand that I am free to withdraw from the study at any time without giving a reason and that this will not disadvantage me in any way.

♦ I agree that should I withdraw from the study, the data collected up to that point may be used by the researcher for the purposes described in the information sheet.

♦ I understand that my name and school and children’s names will not be revealed throughout the process or in the thesis or any other publication arising from the study.

Name (please print) …………………………………………………………………………..

Signed ………………………………………………………………………………………

Date ………………………………………………………………………………………
Appendix F: Parent/carer information sheet and consent form

(Names of the class teacher and Head Teacher have been redacted)

Dear Parent/Carer

I would like to ask your permission for your child to take part in a small research project. My name is Nancy Barclay and I am a mathematics education tutor and doctoral researcher at the University of Brighton. The focus for this project is to explore how teachers can support children to work in pairs to develop their mathematical reasoning. Reasoning is an important part of mathematics learning in the new National Curriculum. Both Mr Bateman and Mrs King are happy for children in the class to take part. The project is taking place in three schools altogether.

This project will take place between October 2014 and June 2015. I will visit your child’s classroom five times to watch the mathematics lessons and to record the detail of what children say and do as they work at their activities. In the visits I will:

1. Watch the lessons and audio record the talk that is part of the normal teaching and learning when the teacher is talking to the class altogether.
2. Video some children as they work at their tables. I wish to use video because when children are working with resources, discussing and recording in their books, a video of their activity will give me more detail than I can get by just watching and making notes.

Not all children will be part of the video group on each day. This will depend on how Mr Bateman has chosen to organise the class and I will not influence this in any way. This means that during the project many different children may be videoed but not necessarily all children.

I would like to ask your consent for your child to take part in this project. To give this consent, please sign and return the attached sheet to Mr Bateman by Wednesday 8th October.

Some important points

• The names of children and the school will not be used in the report or any other publication arising from the study
• If you give your consent your child still has the right to say that s/he does not want to be observed or to talk to me at any time during the process without giving a reason
• The audio and video recordings will be kept securely during the research process and destroyed once the research is completed. Some short sections of video will be viewed only by the teachers taking part in the project, but no copies will be made and no other person will see this data
• I am a qualified teacher and have a valid CRB certificate

If you have any questions that you would like to ask before giving your consent please contact Mr Bateman at school or me, using either phone or email as below

Many thanks

Nancy Barclay
Contact details:
Nancy Barclay, University of Brighton, email n.barclay@brighton.ac.uk; tel 01273 643404

Research supervisors:
Dr. Nadia Edmond: n.edmond@brighton.ac.uk; Dr Els De Geest: enfdegeest@gmail.com
Consent Form

Name of researcher: Nancy Barclay, University of Brighton

Focus of research: supporting the development of children’s mathematical reasoning

◆ I agree that my child may take part in this project which is to explore how children develop reasoning in maths lessons

◆ I have read the information sheet and I understand what is involved.

◆ I understand that Nancy Barclay will watch some maths lessons in my child’s class and may ask my child if they are willing to be video recorded by Nancy Barclay whilst they are working with their maths group

◆ I understand that all audio and video recordings will be kept securely by Nancy Barclay and destroyed once the enquiry is completed.

◆ I understand that short sections of video will be viewed by the teachers in the research project to see how the project is going but that no other person will have access to any other of the data

◆ I understand that my child’s name and school will not be revealed throughout the process or in the report.

◆ I understand that my child is free to withdraw from the study at any time without giving a reason and that this will not disadvantage him/her in any way but that data collected up to that point may still be used.

Name (please print) …………………………………………………………………………………………….

Signed …………………………………………………………………………………………………………

Date …………………………………………………………………………………………………………..
Appendix G Pupil, school and teacher profiles

Pupils

Pupil profiles are included here for the four lower attainers whose data were used in analysis. As noted in chapter 4, some sequences were removed from the data set. Profiles are thus not included for pupils for whom no data is used in this thesis

<table>
<thead>
<tr>
<th>Lower attainer</th>
<th>Higher attainer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mia</strong></td>
<td><strong>Ruby</strong></td>
</tr>
<tr>
<td>Allows others to dominate discussion but is willing to share her thinking when the opportunity arises. Shows conceptual understanding but has weak knowledge and use of procedure. Struggles to recall multiplication facts</td>
<td>Confident in her own thinking and ability to engage in problem solving. Is methodical with recording and able to explain her thinking well. Quick to learn new methods and strategies and focuses well in mathematics lessons.</td>
</tr>
<tr>
<td><strong>Joe</strong></td>
<td><strong>Ash</strong></td>
</tr>
<tr>
<td>Confident to tackle simple problems mentally. Often uses mental methods to tackle more complex problems; makes frequent calculation errors. Struggles with recall of addition and multiplication facts. Sometimes doesn’t persist with problem solving and is relatively easily distracted. Enjoys paired work and always willing to have a go.</td>
<td>Confident to contribute in class discussions; confident in tackling problems and will engage readily in problem solving. Is quick to identify an approach to work with. Tends to work quickly and sometimes makes errors through unsystematic or unclear recording or not thinking through the steps sufficiently.</td>
</tr>
<tr>
<td><strong>Daisy</strong></td>
<td><strong>Alice</strong></td>
</tr>
<tr>
<td>Enjoys working with a partner but will let people ‘carry’ her sometimes. Will express opinions and correct others when she is clear a mistake has been made. However, makes frequent errors and is hesitant to apply systematic processes. Looks to others for ideas</td>
<td>Will readily share thoughts in class and with a partner. Is patient and methodical in approaching a task, but sometimes makes errors because recording has been rushed.</td>
</tr>
<tr>
<td><strong>Rose</strong></td>
<td><strong>Poppy</strong></td>
</tr>
<tr>
<td>Lacks confidence in her own capacity to make sense of a problem and seeks help from others or from the teacher frequently. Is able to follow a process and can work carefully and with enthusiasm when she understands what is required of her. Will sustain concentration but equally needs reassurance to do so.</td>
<td>Confident of her own mathematical capacity. Confident to lead a group activity and quickly identifies ways to work. Does not contribute strongly to class discussion.</td>
</tr>
<tr>
<td><strong>Jess</strong></td>
<td><strong>Hannah</strong></td>
</tr>
<tr>
<td>Very confident, high achieving in tests. Quick to spot patterns and express generality. Takes time to think about things carefully. Works well with others and takes care to explain her thinking to a partner</td>
<td>High achieving and methodical in her approach. Confident to share thinking in class and works well with others. Confident to lead activity. Able to identify generality.</td>
</tr>
</tbody>
</table>
SCHOOLs

School A is a large urban junior school with 3 classes per year group and over 500 pupils on role. It is graded outstanding by Ofsted. The proportion of pupils known to be eligible for free school meals is less than average as is the percentage of pupils from ethnic minority backgrounds; the percentage of pupils with SEND is average.

School B is a larger than average two form entry urban primary school with over 400 pupils on roll. The proportion of pupils known to be eligible for free school meals is well above the national average as is the proportion of pupils with SEND. The proportion of pupils from ethnic minority backgrounds is lower than the national average. It is rated as good by Ofsted.

School C is a single form entry primary school with just over 200 pupils on roll. It was identified as requiring improvement by Ofsted in the period in which this research took place. The proportion of pupils known to be eligible for free school meals is well below the national average as is the proportion of pupils from ethnic minorities and of those who have SEND.

TEACHERs

Teacher 1 has been teaching since 2012 and is experienced in KS2. Teacher 1 completed the MaST programme in 2014.

Teacher 2 has been teaching since 2000 and is experienced in KS1 and KS2. Teacher 2 completed the MaST programme in summer 2013.

Teacher 3 has been teaching since 2007 and is experienced in KS2. Teacher 3 completed the MaST programme in summer 2013.