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Economic Periodic Orbits: A Theory of Exponential Asymptotic Stability

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Abstract

This paper establishes the conditions for existence, uniqueness, and exponentially asymptotically stability of periodic orbits of a dynamical system defined by a set of ordinary differential equations with discontinuous right-hand side. Moreover, a formula for the basin of attraction is provided. These results equip economists with a set of tools, which will allow them to generate new analytic results.

Keywords: Economic Equilibrium, Dynamical Systems, Exponential Asymptotic Stability

1 Introduction

Filippov introduces a solution concept for differential equations with discontinuous right hand side [2]. Such equations frequently appear in economic modelling [7], [4], [1], [5], [6]. Despite the importance of these models in economic policy analysis, economic equilibrium analysis, however, rarely goes beyond existence and uniqueness results. The aim of this paper is to improve on this and to therefore establish the conditions of exponentially asymptotically stability of non-smooth solutions. Moreover, the theory also provides a

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formula for the characterization of the set of initial conditions of these solutions. For that purpose, we consider a nonsmooth dynamical system defined by an autonomous ordinary differential equation

$$\dot{x} = f(x)$$

where f is a discontinuous function at $x_2 = 0$ and $x \in \mathbb{R}^2$ [8]. The discontinuity of $f \in C^1(\mathbb{R}^2 \backslash (\mathbb{R} \times \{0\}), \mathbb{R}^2)$ implies that the phase space $\mathbb{X} = \mathbb{R}^2$ is divided into subspaces $\mathbb{X} = \mathbb{X}^+ \cup \mathbb{X}^0 \cup \mathbb{X}^-$, where $\mathbb{X}^+ = \{x \in \mathbb{R}^2 : x_2 > 0\}$, $\mathbb{X}^- = \{x \in \mathbb{R}^2 : x_2 < 0\}$, and $\mathbb{X}^0 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. By defining $f := f^{\pm}$ where $f(x) = f^+(x)$ if $x \in \mathbb{X}^+$, and $f(x) = f^-(x)$ if $x \in \mathbb{X}^-$, we have

$$\dot{x} = f(x) = \begin{cases} f^+(x) & \text{if } x \in \mathbb{X}^+ \\ f^-(x) & \text{if } x \in \mathbb{X}^-. \end{cases} \tag{1}$$

An initial value condition for this system at time $t = 0$ is given by $x(0) = x_0 \in \mathbb{X}$. We restrict ourselves to a set of assumptions which according to a sequence of results by Filippov [2] guarantees global existence, uniqueness, and continuous dependence on the initial condition of solutions of the differential equation (1).

Assumption 1 *Consider equation (1). We assume $f \in C^1(\mathbb{R}^2 \backslash (\mathbb{R} \times \{0\}), \mathbb{X})$. Each function $f^{\pm}(x)$ with $x \in \mathbb{X}^+$ or $x \in \mathbb{X}^-$ can be extended to a continuous function up to $x \in \mathbb{X}^0$. Each function $Df^{\pm}(x)$ with $x \in \mathbb{X}^+$ or $x \in \mathbb{X}^-$ can be extended to a continuous function up to $x \in \mathbb{X}^0$. $f^+(x_1, 0) - f^-(x_1, 0)$ is a C^1-function with respect to x_1. For all $(x_1, 0) \in \mathbb{X}^0$ it holds that $f_2^+(x_1, 0) \cdot f_2^-(x_1, 0) > 0$.*

The assumption $f_2^+(x_1, 0) \cdot f_2^-(x_1, 0) > 0$ for all $(x_1, 0) \in \mathbb{X}^0$ states that $f_2^+, f_2^- < 0$, or $f_2^+, f_2^- > 0$. This assumption excludes all sliding phenomena on the $\mathbb{X}^0$ manifold and shall be relaxed in future work. It implies a discontinuity in $+/-$ direction if both $f_2^+, f_2^- < 0$ or a discontinuity in $-/+$ direction if both $f_2^+, f_2^- > 0$. Let the flow of the system given by (1) be defined by $S_t(x_0) := (x_1(t), x_2(t)) \in \mathbb{X}$, where $(x_1(t), x_2(t)) \in \mathbb{X}$ is its solution with initial value $((x_1(0), x_2(0)) = x_0$. Hence the flow $S_t x_0$ maps the initial point x_0 at time $t = 0$ to a point $x(t)$ at time $t \geq 0$. An adjacent trajectory is defined by $S_\theta(x_0 + \eta) := (y_1(\theta), y_2(\theta)) \in \mathbb{X}$, where $(y_1(\theta), y_2(\theta)) \in \mathbb{X}$, and $\|\eta\| > 0$ and time $\theta \geq 0$. Let $K \subseteq \mathbb{X}$ be positively invariant if $S_t x_0 \in K$ for all $t \geq 0$ and all $x_0 \in K$. A periodic orbit Ω of the system (1) is a set defined by $\Omega := \{S_t(x_0) : t \in [0, T]$, such that $S_T(x_0) = x_0\} \subset \mathbb{X}$, with minimal period $T > 0$. Let $K \subseteq \mathbb{X}$ and $K \neq \varnothing$ be a compact, connected and positively invariant set which contains no equilibria. Moreover, set $K^+ := K \cap \{x \in \mathbb{R}^2 : x_2 > 0\}$ and $K^- := K \cap \{x \in \mathbb{R}^2 : x_2 < 0\}$. We now define a neighborhood $A(\Omega)$ of Ω consisting of

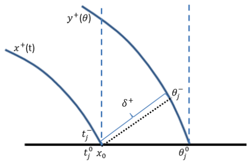

Figure 1: Case I.

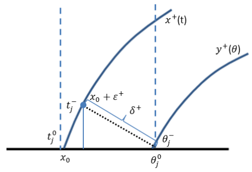

Figure 2: Case II.

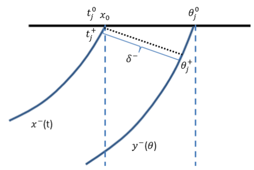

Figure 3: Case III.

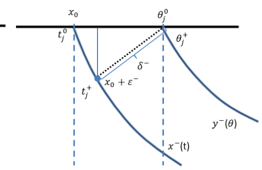

Figure 4: Case IV.

a set of points x_0 in $\mathbb{X}$ such that the distance between $S_t(x_0)$ and Ω vanishes as $t \to \infty$. The basin of attraction $A(\Omega)$ of an exponentially asymptotically stable orbit Ω is the set defined by $A(\Omega) := \{x_0 \in \mathbb{X} : dist(S_t x_0, \Omega) \xrightarrow{t\to\infty} 0\}$.

Theorem 2 *Let assumption 1 hold, and let $\emptyset \neq K \subset \mathbb{R}^2$ be a compact, connected and positively invariant set with $f^{\pm}(x) \neq 0$ for all $x \in K^{\pm}$. Moreover, assume that $W^{\pm} : \mathbb{X}^{\pm} \to \mathbb{R}$ are continuous functions and let the orbital derivatives $(W^{\pm})'$ exist and be continuous functions in $\mathbb{X}^{\pm}$ and continuously extendable up to $\mathbb{X}_0$. We set $K^0 := \{x \in \ K : x_2 = 0\}$. Let the following conditions hold:*

1. $L_{W^{\pm}(x)} := \max_{\|v^{\pm}\| = e^{-W^{\pm}(x)}, v^{\pm} \perp f^{\pm}(x)} L_{W^{\pm}}(x, v^{\pm}) \leq -\nu < 0$
   $L_{W^{\pm}}(x, v^{\pm}) := e^{2W^{\pm}(x)} \left\{ (v^{\pm})^T \left[Df^{\pm}(x) \right] v^{\pm} + \langle \nabla W^{\pm}(x), f^{\pm}(x) \rangle \|v^{\pm}\|^2 \right\}$
   for all $x \in K^{\pm}$.
2. $\frac{f_2^{\mp}(x)}{f_2^{\pm}(x)} \cdot \frac{\sqrt{\left(f_1^{\mp}(x)\right)^2 + \left(f_2^{\mp}(x)\right)^2}}{\sqrt{\left(f_1^{\mp}(x)\right)^2 + \left(f_2^{\mp}(x)\right)^2}} e^{W^{\mp}(x) - W^{\pm}(x)} < 1$
   for all $x \in K^0$ with $f_2^{\pm}(x) < 0$, $f_2^{\mp}(x) < 0$.

Then there is one and only one periodic orbit $\Omega \subset K$. Moreover, Ω is exponentially asymptotically stable with exponent $-\nu < 0$ and for its basin of attraction the inclusion $K \subset A(\Omega)$ holds.

Definition 1 *For all $(\delta_1, \delta_2) > 0$ define $(\varepsilon_1, \varepsilon_2) > 0$ such that $\|f^{\pm}(e_1, e_2) - f^{\pm}(x_1, 0)\| < \|\delta\|$ for all $(e_1, e_2) < (\varepsilon_1, \varepsilon_2)$ holds.*

Proof. The strategy of the proof is to provide a local contraction metric at any point of discontinuity of f. Four types of discontinuity arise in our model. These are all symmetric and diametrically opposed to case I. For smooth solutions we already have a complete contraction theory [3].

Consider a distance function $A^+ : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$A^{\pm}(t) := \sqrt{\left(\left(S^{\pm}_{\mathcal{T}}(x+\eta) - S^{\pm}_t x\right)^T e^{2W^{\pm}(S^{\pm}_t x)} \left(S^{\pm}_{T_x^{x+\eta}(t)}(x+\eta) - S^{\pm}_t x\right)\right)}$$

Let $K^0 := \{x \in \mathbb{X} : \mathbb{X} \backslash \{\mathbb{R} \times \{0\}\}\}$. By conditions of theorem 2, K^0 is a non-empty and compact set which contains no equilibrium. Let $G > 0$ be a constant such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_2^+(x_0) = -c \leq -G$,

$$-G := \max_{x_0 \in K^0, f_2^+(x_0) < 0} f_2^+(x_0)$$

Let $\varepsilon_2 \leq \frac{G}{2}$. Also let there be a constant $D > 0$ such that for all $x_0 \in K^0$ satisfying $f_2^+(x_0) < 0$, $f_1^+(S_t^+ x_0) = d \leq D$,

$$D := \max_{x_0 \in K^0, f_2^+(x_0) < 0} |f_1^+(x_0)|$$

Let $\varepsilon_1 \leq \frac{D}{2}$. Given $(\varepsilon_1, \varepsilon_2) > 0$ there is a $(b_1, b_2) > 0$ so that we can construct a box in the positive phase space with center x_0 by

$$B_{(x_0)}^+ := \left\{(y_1, y_2) \in K \cap [x_0^1 - b_1, x_0^1 + b_1] \times [0, b_2]\right\}$$

such that for all $x \in B_{(x_0)}^+$

$$\begin{aligned} -\frac{D}{2} \leq \mid d \mid -\varepsilon_1 \leq -d - \varepsilon_1 \quad &\leq \\ f_1^+(y) \quad &\leq \quad d + \varepsilon_1 \leq \mid d \mid + \varepsilon_1 \leq \frac{3D}{2} \qquad (2) \\ -\frac{3G}{2} \leq -c - \varepsilon_2 \quad \leq \quad f_2^+(y) &\leq -c + \varepsilon_2 \leq -\frac{G}{2}. \qquad (3) \end{aligned}$$

Consider a solution $y_2^+(\theta)$ with $y_2^+(\theta_j^-) \in B_{(x_0)}^+$ and $y_2^+(\theta_j^-) > 0$. Now, we want to show that there is a time θ^* such that the solution $y_2^+(\theta^*) = 0$. A solution $y_2^+(\tau) > 0$ decreases to $y_2^+(\theta^*) = 0$ since $f_2^+(y_2^+(\tau)) \leq \frac{-G}{2} < 0$ for all $y_2^+(\tau) \in B_{(x_0)}^+$ with $f_2^+(y_2^+(\tau)) \in [-c - \varepsilon_2, -c + \varepsilon_2]$ and $\tau \in [\theta_j^-, \theta^*]$. We have

$$y_2^+(\theta) - y_2^+(\theta_j^-) = \int_{\theta_j^-}^{\theta_j^0} f_2^+(y_2^+(\tau)) d\tau.$$

Define $\delta_2 = y_2^+(\theta_j^-)$ and $e_2 = \frac{1}{\tau^*} \int_{\theta_j^-}^{\theta^*} [f_2^+(y_2^+(\tau)) + c] d\tau$. Then since $y_2^+(\theta^*) = 0$ and $|e_2| \leq \varepsilon_2$ we have by equation (3) that

$$\frac{\delta_2}{c - e_2} = \theta^*. \qquad (4)$$

Hence

$$\frac{\delta_2}{c + |e_2|} \leq \theta^*.$$

Thus there is a time $\theta^* \in [\frac{\delta_2}{c+\varepsilon_2}, \frac{\delta_2}{c-\varepsilon_2}]$ such that $y_2^+(\theta^*) = 0$. We have shown that for frozen time $t_j^- = t_j^+$ with $j \in \{2n\}$ and $n \in \mathbb{N}_0$ of solution $x^+(t)$ there is a time $\theta^* :=\mid \theta_j^0 - \theta_j^- \mid$ of solution $y^+(\theta)$ such that $\frac{\delta_2}{c-\varepsilon_2} \geq \theta^*$. Now, we use θ^* in (4) in order to determine $y_1^+(\theta^*)$. We have

$$y_1^+(\theta^*) \quad = \quad y_1^+(\theta_j^-) + \int_{\theta_j^-}^{\theta^*} f_1^+(y^+(\tau))d\tau$$

Define $\delta_1 = y_1^+(\theta_j^-) - x_1^+(t_j^0)$ and $e_1 = \frac{1}{\tau^*}\int_{\theta_j^-}^{\theta^*}[f_1^+(y^+(\tau)) - d]d\tau$, and since $|e_1| \leq \varepsilon_1$ we have

$$= \quad \delta_1 + \theta^* \cdot [d + e_1]$$

and by equation (4) we obtain

$$= \quad \delta_1 + \frac{\delta_2}{c - e_2} \cdot [d + e_1].$$

Using $\delta_1 f_1^+(x_0) + \delta_2 f_2^+(x_0) = 0$ yields

$$\begin{aligned} &= \quad \delta_1 - \left(\frac{\delta_1 f_1^+(x_0)}{(-f_2^+(x_0) - e_2) f_2^+(x_0)} \right) \cdot [f_1^+(x_0) + e_1] \\ &= \quad \delta_1 \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + e_1 \cdot f_1^+(x_0) + e_2 \cdot f_2^+(x_0)}{(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)} \right]. \end{aligned}$$

We have the following bounds

$$\begin{aligned} \mid \delta_1 \mid \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 - \varepsilon_1 \mid f_1^+(x_0) \mid - \varepsilon_2 \mid f_2^+(x_0) \mid}{(f_2^+(x_0))^2 + \varepsilon_2 \mid f_2^+(x_0) \mid} \right] \quad &\leq \\ |y_1^+(\theta^*) - x_1^+(t_j^0)| \quad &\leq \\ \mid \delta_1 \mid \left[\frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + \varepsilon_1 \mid f_1^+(x_0) \mid + \varepsilon_2 \mid f_2^+(x_0) \mid}{(f_2^+(x_0))^2 - \varepsilon_2 \mid f_2^+(x_0) \mid} \right] & \end{aligned}$$

We need to verify the bounds. This only requires to check that

$$(f_2^+(x_0))^2 - \varepsilon_2 |f_2^+(x_0)| \geq E > 0.$$

We have $|f_2^+(x_0)| = c$. Hence from $(f_2^+(x_0))^2 - \varepsilon_2 |f_2^+(x_0)|$ we obtain

$$\begin{aligned} c(c - \varepsilon_2) \quad &\geq \quad c(c - \frac{G}{2}) \\ &\geq \quad G(G - \frac{G}{2}) = \frac{G^2}{2} > 0. \end{aligned}$$

as required.

We consider $|y_1^+(\theta^*) - x_1^+(t_j^0)|$ and define

$$r_b :=\mid \delta_1 \mid \cdot \left| \frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + e_1 \cdot f_1^+(x_0) + e_2 \cdot f_2^+(x_0)}{(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)} \right| = |y_1^+(\theta^*) - x_1^+(t_j^0)|.$$

The calculation for $A^+(t_0)$ follows from

$$\begin{aligned} A^+(t_0) &= \sqrt{\delta_1^2 + \delta_2^2} \\ &= \sqrt{\delta_1^2 + \delta_1^2 \frac{\left(f_1^+(x_0)\right)^2}{\left(f_2^+(x_0)\right)^2}} \\ &= \mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}{\mid f_2^+(x_0) \mid} \end{aligned}$$

We now calculate $\frac{A^+(t_0)}{r_b}$. We have

$$\begin{aligned} \frac{A^+(t_0)}{r_b} &= \frac{\mid \delta_1 \mid \frac{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}{|f_2^+(x_0)|}}{\mid \delta_1 \mid \left| \frac{(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + f_1^+(x_0) \cdot e_1 + e_2 \cdot f_2^+(x_0)}{(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)} \right|} \\ &= \frac{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}{|f_2^+(x_0)|} \cdot \frac{|(f_2^+(x_0))^2 + e_2 \cdot f_2^+(x_0)|}{|(f_1^+(x_0))^2 + (f_2^+(x_0))^2 + f_1^+(x_0) \cdot e_1 + e_2 \cdot f_2^+(x_0)|} \\ &= \frac{|f_2^+(x_0) + e_2|}{\left| \sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2} + \frac{f_1^+(x_0) \cdot e_1 + e_2 \cdot f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}} \right|}. \end{aligned}$$

Let

$$f^+(e_1, e_2) := \frac{|f_2^+(x_0) + e_2|}{\left| \sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2} + \frac{f_1^+(x_0) \cdot e_1 + e_2 \cdot f_2^+(x_0)}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}} \right|} \tag{5}$$

with bounds

$$\begin{aligned} \frac{\mid f_2^+(x_0) \mid - \varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2} + \frac{\varepsilon_1 \cdot |f_1^+(x_0)| + \varepsilon_2 \cdot |f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}} &\leq \\ f^+(e) &\leq \\ \frac{\mid f_2^+(x_0) \mid + \varepsilon_2}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2} + \frac{-\varepsilon_1 \cdot |f_1^+(x_0)| - \varepsilon_2 \cdot |f_2^+(x_0)|}{\sqrt{\left(f_1^+(x_0)\right)^2 + \left(f_2^+(x_0)\right)^2}}} & \end{aligned}$$

We need to show that the denominator E of $f^+(e_1, e_2)$ is strictly positive. This only requires to check that

$$(f_2^+(x_0))^2 - \varepsilon_2|f_2^+(x_0)| + (f_1^+(x_0))^2 - \varepsilon_1|f_1^+(x_0)| \geq E > 0.$$

This follows from a similar process as above. Similarly to the method outlined here, Stiefenhofer [8] shows that we can also establish the conditions for cases II-IV. Hence $\frac{I}{III}$ and $\frac{II}{IV}$ yield the discontinuity conditions of theorem (2) as requested. ■

2 Conclusion

Differential equations are ubiquitous in economics. However, since Filippov [2], little progress beyond existence and uniqueness of equilibrium has occurred in the economics literature. The purpose of this paper is to provide a method, which allows economists to establish analytic results on exponentially asymptotically stability of non-smooth periodic orbits, and to characterize their basin of attraction. It is anticipated that such results are desirable in economics for the purpose of economic policy analysis. The advantage of our method is that it is possible to establish exponentially asymptotically stability of periodic orbits without the need of explicitly solving the system of ordinary differential equations with discontinuous right hand side.

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