A simple model for puffing/micro-explosions in water-fuel emulsion droplets

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A R T I C L E   I N F O

Article info
Received 3 August 2018
Received in revised form 26 October 2018
Accepted 13 November 2018

Keywords:
n-Dodecane
Droplets
Micro-explosions
Puffing
Heat conduction equation

A B S T R A C T

A new simple model for the puffing/micro-explosion of water-fuel emulsion droplets is suggested. The model is based on the assumption that a spherical water sub-droplet is located in the centre of a larger fuel droplet. The fuel is approximated by n-dodecane. The fuel droplet surface temperature is assumed to be fixed, and fuel evaporation is ignored. The heat conduction equation is solved inside this composite droplet with the Dirichlet boundary condition at the surface of the fuel droplet. The time instant when the temperature at the interface between water and fuel reaches the boiling temperature of water is associated with the start of the puffing process leading to micro-explosion. This time is referred to as the time to puffing, or micro-explosion delay time. When the fuel surface temperature is equal to the boiling temperature of fuel then this time is expected to be the shortest of the possible times. The predictions of the model are shown to be in agreement with available experimental data. The model predicts an increase in this time with increase in the fuel droplet size for fixed water and fuel mass fractions, in agreement with observations.

1. Introduction

The potential for water-in-diesel emulsions to simultaneously reduce NOx and particulate emissions has generated significant attention. Water-emulsified fuels could be an efficient approach to the reduction of exhaust emissions while improving engine performance, without requiring any engine modification. Diesel and biodiesel-Diesel emulsions were found to increase brake thermal performance, without requiring any engine modification. Diesel and biodiesel-Diesel emulsions were found to increase brake thermal efficiency by approximately 6%, with a 30% reduction in NOx and smoke, and a 70% reduction in unburnt hydrocarbons [1]. An increase in brake specific fuel consumption (BSFC) is obtained due to the lower calorific value of the emulsions [2] as water displaces fuel. The increase in water concentration results in longer ignition delays due to the lower volatility and higher viscosity of water [3]. The reduction in NOx seems to be related to reduction in in-cylinder temperature obtained through the evaporation of the emulsion's water [4,5]. The increased formation of hydroxyl radicals reduces soot emissions by increasing the oxidation rate [6]. Research shows that the properties of the dispersed water droplets significantly affect NOx emissions [7] and engine efficiency [8], with smaller dispersed droplet sizes leading to improvements in combustion and emissions. There is also some indication that the combustion process is improved through an increased secondary atomization [9,10], leading to better air-fuel mixing [11]. It is important to note that the fuel injection hardware can significantly affect the emulsion properties, with the dispersed droplet sizes reducing significantly after the emulsion is compressed by the high-pressure fuel pump, and again after being injected through the nozzle orifices [12].

A widely used mechanism to improve fuel mixing is through the phenomena of puffing and micro-explosion in emulsion droplets, which could promote the secondary atomization process through a rapid break-up of the parent droplets due to the different volatility of the fuel (continuous phase) and water (dispersed phase). Puffing is the partial ejection of some of the dispersed water out of an emulsion droplet [14], while micro-explosion is the complete break-up of the parent droplet. The roles of puffing and micro-explosions in water-in-fuel emulsion droplets in internal combustion engines have been widely discussed in the literature (e.g. [13–16] and the references therein). Both phenomena enhance the effective fuel droplet size distribution, air-fuel mixing, and ultimately fuel efficiency [17]. Observations show that these events are influenced by water concentration, surfactant type, and dispersed water droplet size and temperature [18,19]. Some researchers observed that the intensity of micro-explosions...
increased with water content [20,21] as well as with the droplet size of the dispersed water phase.

It seems that Ivanov and Nefedov [22] were the first to draw attention to micro-explosions as a mechanism for accelerated evaporation of emulsion fuel droplets where water was the volatile component. Since this pioneering paper a number of models for micro-explosions have been suggested, starting with relatively simple ones [23,24] and ending with the recently suggested most advanced models [14–16].

Although the models developed in [14–16] are undoubtedly very useful for understanding the physical background of the processes involved in puffing/micro-explosions in droplets, the usefulness of these models for engineering applications is less obvious. Engineers dealing with puffing/micro-explosions are interested primarily in the integral characteristics of the process, such as the puffing/micro-explosion delay times, rather than in the details of the processes inside the deforming and exploding droplet. In our paper, the results of the development of a simple model specifically focused on the prediction of these delay times are presented and discussed.

2. Description of the new model

The model is based on the assumption that a spherical water sub-droplet is located in the centre of a fuel droplet as shown in Fig. 1. In this figure, \( R_w \) and \( R_d \) are the radii of the water sub-droplet and fuel droplet, respectively. \( T_w \) and \( T_s \) in this figure are the temperature at the interface between water and fuel and the temperature at the surface of the fuel droplet, respectively. The fuel is approximated by n-dodecane. The temperature \( T_s \) is assumed to be fixed, and fuel evaporation is ignored. The droplet is assumed to be stationary and penetration of heat from the surface of the droplet to its interior is described by a one-dimensional transient heat conduction equation. This equation is solved analytically inside this composite droplet with the Dirichlet boundary condition at its surface, and the solution leads to a time-dependent distribution of temperature inside the droplet. If \( T_s \) is greater than the initial temperature inside the droplet, assumed to be homogeneous, then the temperature between the centre of the droplet and its surface will increase. If \( T_s \) is greater than the boiling temperature of water, at a certain time instant \( T_w \) is expected to reach this boiling temperature. This time instant is associated with the start of the puffing process leading to micro-explosion. This time is referred to as the time to puffing, or micro-explosion delay time. The maximal value of \( T_s \) is assumed to be equal to the boiling temperature of n-dodecane. In this case, the micro-explosion delay time is expected to be minimal.

Note that a number of authors drew attention to the fact that the start of the puffing processes can be associated with the time instant when \( T_w \) reaches the nucleation rather than boiling temperature [25]. The former temperature is defined as the mean temperature in the range in which the nucleation rate changes from a negligible value to a very large value [25]. We have not, however, found reliable values of this temperature for the conditions of the experiment which will be used for validation of our model. To be consistent with our other simplifying assumptions of the model we base our analysis on the assumption that puffing starts at the boiling temperature.

The mathematical details of the model are described in the next section. The results of the application of the model to the analysis of experimentally observed puffing/micro-explosions in composite water-n-dodecane droplets are described in Section 4. The main results of the paper are summarised in Section 5.

3. Basic equations and approximations

The variation of the temperature in the water-fuel domain, shown in Fig. 1, is described by the heat conduction equation in the form [26,27]:

\[
\frac{\partial T}{\partial t} = \frac{1}{\gamma} \frac{\partial}{\partial \rho} \left( \gamma \frac{T}{\gamma} \frac{\partial T}{\partial \rho} \right)
\]

**Nomenclature**

- \( a \) function introduced in Solution (A1.8)
- \( b \) weight of functions \( v_n(R) \) (J/(m³ K))
- \( B \) function introduced in Solution (A1.8)
- \( c \) specific heat capacity (J/(kg K))
- \( h \) convective heat transfer coefficient (W/(m² K))
- \( k \) thermal conductivity (W/(m K))
- \( P \) radiation source term in Eq. (1) (K/s)
- \( p_n(t) \) functions introduced in Eq. (8) (K m/s)
- \( R \) distance from the droplet centre (m)
- \( R_{w(d)} \) radius of the water sub-droplet (droplet) (m)
- \( T \) temperature (K)
- \( T_0 \) initial temperature inside the droplet assumed to be equal to the boiling temperature of n-dodecane. In this case, the micro-explosion delay time is expected to be minimal.
- \( T_1 \) is assumed to be equal to the boiling temperature of n-dodecane. In this case, the micro-explosion delay time is expected to be minimal.
- \( u \) function defined by Eq. (9) (K m)
- \( v_n \) eigenfunctions found from Eq. (12)
- \( \|v_n\| \) norm of \( v_n \) with weight \( b \) (J/(m³ K))
- \( \Omega_n(t) \) function introduced in Eq. (A1.4) (K m)
- \( \Theta_n(t) \) function defined by Eq. (10) (K m)
- \( \Omega_n(t) \) function defined by Eq. (10) (K m)
- \( \kappa \) thermal diffusivity (m²/s)
- \( T_s \) is assumed to be fixed, and fuel evaporation is ignored. The droplet is assumed to be stationary and penetration of heat from the surface of the droplet to its interior is described by a one-dimensional transient heat conduction equation. This equation is solved analytically inside this composite droplet with the Dirichlet boundary condition at its surface, and the solution leads to a time-dependent distribution of temperature inside the droplet. If \( T_s \) is greater than the initial temperature inside the droplet, assumed to be homogeneous, then the temperature between the centre of the droplet and its surface will increase. If \( T_s \) is greater than the boiling temperature of water, at a certain time instant \( T_w \) is expected to reach this boiling temperature. This time instant is associated with the start of the puffing process leading to micro-explosion. This time is referred to as the time to puffing, or micro-explosion delay time. The maximal value of \( T_s \) is assumed to be equal to the boiling temperature of n-dodecane. In this case, the micro-explosion delay time is expected to be minimal.

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\[
\frac{\partial T}{\partial t} = \frac{1}{\gamma} \frac{\partial}{\partial \rho} \left( \gamma \frac{T}{\gamma} \frac{\partial T}{\partial \rho} \right)
\]
\[ \frac{\partial T}{\partial t} = \kappa \left( \frac{\partial^2 T}{\partial R^2} + \frac{2 \partial T}{\partial R} \right) + P(t, R), \tag{1} \]

where
\[ \kappa = \begin{cases} \kappa_w = k_w/(C_0 \rho_w) & \text{when } R \leq R_w \\ \kappa_f = k_f/(C_f \rho_f) & \text{when } R_w < R \leq R_d, \end{cases} \tag{2} \]

\( k_w \) is the water (liquid fuel) thermal diffusivity, \( k_w \) is the water (liquid fuel) thermal conductivity, \( C_0 \) is the water (liquid fuel) specific heat capacity, \( \rho_w \) is the water (liquid fuel) density, \( R \) is the distance from the centre of the droplet, \( t \) is time. The source term \( P(t, R) \) takes into account the volumetric droplet heating (e.g., heating due to thermal radiation).

Eq. (1) needs to be solved subject to initial and boundary conditions:
\[ T|_{t=0} = \begin{cases} T_{w0}(R) & \text{when } R \leq R_w \\ T_{f0}(R) & \text{when } R_w < R \leq R_d, \end{cases} \tag{3} \]
\[ T|_{R=R_w} = T|_{R=R_d}, \]
\[ h(T_g - T(R_d)) = k_f \frac{\partial T}{\partial R} \bigg|_{R=R_d}. \tag{5} \]

where \( h \) is the convection heat transfer coefficient, describing the heating of the droplet. To ensure consistency between Conditions (3) and (4) we assume that \( T_{w0}(R_w) = T_{f0}(R_w). \)

Expression (5) is the well-known Robin boundary condition widely used for the analysis of droplet heating and evaporation [28]. Let us rewrite this equation as:
\[ T(R_d) = T_s = T_g - k_f \frac{\partial T}{\partial R} \bigg|_{R=R_d}. \tag{6} \]

If the solution to Eq. (1) is performed at a small time step, we could make an assumption that \( T_R = T_{g0} = \text{const} \) and \( h = \text{const} \). Moreover, we could take the value of \( \frac{\partial T}{\partial R} \bigg|_{R=R_d} \) in (6) from the initial condition for the first time step and from the results of calculation for the previous time steps for all the following steps. This would allow us to assume that the right hand side of Eq. (6) is constant and reduce the Robin boundary condition (5) to the Dirichlet boundary condition:
\[ T(R_d) = T_s. \tag{7} \]

To be consistent with the previous simplifying assumptions of the model, we will make several further simplifying assumptions.

1. The temperature of the droplet surface is assumed to be constant during the whole period of droplet heating (not just during individual time steps). This would have been a natural assumption for the times after the droplet surface temperature reaches the boiling point. In the general case, the assumption that the droplet surface temperature is equal to the boiling temperature of the liquid fuel would allow us to predict the minimal time to puffing and micro-explosion delay time. More realistic values of this temperature are expected to be below the liquid fuel boiling temperature and this will be taken into account in the analysis of our results.

2. The effects of droplet evaporation and swelling are ignored. This means that we assume that puffing/micro-explosion occurs before the droplet evaporation becomes significant, which is consistent with experimental observations. The effects of swelling are anticipated to be small and can be safely ignored in the analysis remembering the crudeness of the whole model.

3. We assume that both \( T_w \) and \( T_0 \) do not depend on \( R \). Eventually they will be assumed equal to \( T_{g0}. \)

4. The effect of the source term (thermal radiation) is taken into account when deriving the formulae, but eventually it will be ignored in our analysis. This approach would allow us to generalise the model in the future to the case where the contribution of thermal radiation is expected to be significant.

5. All thermodynamic and transport properties (liquid thermal conductivities, heat capacities and densities) are taken at the initial droplet temperature. These properties are expected to be relatively weak functions of temperature; ignoring this dependence is considered to be consistent with other simplifying assumptions of the model.

These assumptions allow us to generalise the analysis previously described in [29,30] and derive the analytical solution to Eq. (1), subject to initial and boundary conditions (3), (4) and (7) in the form (see Appendix A for further details):
\[ T(R, t) = T_s + \frac{1}{R} \sum_{n=1}^{\infty} \exp \left( -j_n^2 \frac{t}{t_d} \right) \left[ \Theta_{n1} + \Theta_{n2} \right] \]
\[ + \int_0^{t_d} \exp \left( -j_n^2 (t - \tau) \right) \frac{\partial}{\partial \tau} \left[ \nu_1(R, \tau) \right] d\tau, \tag{8} \]

where
\[ \Theta_{n1} = \frac{T_0 c_w \rho_w}{\left| \nu_0 \right|^2 (j_n a_n R_w) \cot (j_n a_n R_w) - 1}, \]
\[ \Theta_{n2} = \frac{T_0 c_f \rho_f}{\left| \nu_0 \right|^2 (j_n a_d R_d) \cot (j_n a_d (R_d - R_w)) - 1} \]
\[ - \frac{a_n a_d}{\sin (j_n a_d (R_d - R_w))} \]
\[ T_0 = T_s - T_{w0} = T_s - T_{f0}. \tag{11} \]
\[ \nu_1(R) = \begin{cases} \frac{\sin (j_n a_n R_w)}{\sin (j_n a_d (R_d - R_w))} & \text{when } R < R_w \\ \frac{\sin (j_n a_d (R_d - R_w))}{\sin (j_n a_d (R_d - R_w))} & \text{when } R_w < R \leq R_d, \end{cases} \tag{12} \]
\[ \left| \nu_0 \right|^2 = \frac{c_w \rho_w R_w}{2 \sin^2 (j_n a_n R_w)} + \frac{c_f \rho_f (R_d - R_w)}{2 \sin^2 (j_n a_d (R_d - R_w))} - \frac{k_w - k_f}{2 R_w^2 \rho_w}, \tag{13} \]
\[ p_n(t) \]
\[ \left| \nu_0 \right|^2 \int_0^{R_w} R P(t, R) v_n(R) dR + \frac{c_f \rho_f (R_d - R_w)}{\left| \nu_0 \right|^2} \int_0^{R_d} R v_n(R) P(t, R) dR. \]

A countable set of positive eigenvalues \( j_n \) is found from the solution to the equation:
\[ \sqrt{k_w c_w \rho_w \cot (j_n a_n R_w)} - \sqrt{k_f c_f \rho_f \cot (j_n a_d (R_d - R_w))} = \frac{k_w - k_f}{R_w^2 \rho_w}. \tag{13} \]

These are arranged in ascending order \( 0 < j_1 < j_2 < \ldots \)
\[ a_w = \sqrt{\frac{c_w \rho_w}{k_w}}, \quad a_f = \sqrt{\frac{c_f \rho_f}{k_f}}. \]

In Expression (12) sign ‘−’ was chosen in order to obtain a physically meaningful solution when \( t < T_s \).

4. Analysis

Experiments were conducted in a heated vessel (700 K) at atmospheric pressure, with optical access to observe the sprayed emulsion. The experimental setup is described in [31]. The time
to puffing was measured using a high-speed camera fitted with a long-distance microscope, the images from which were processed to measure the initial droplet diameters and the time to puffing. In order to account for the significant effects of the fuel injection system on the properties of the dispersed phase [12], we first ran the emulsion through a high-pressure pump and injector. This enabled us to collect and analyse the sprayed samples’ dispersed water droplets before injecting the emulsion into the optical vessel using a low-pressure micro-syringe.

The experiments were performed with a resolution of 256 × 800 pixels², frame rate of 12,000 frames per second, and exposure time of 2 µs. The image scale factor was 0.0185 mm/pixel. Hence, the uncertainty for the time to puffing was linked to the detection of the frames for droplet creation and start of puffing, which corresponds to ±167 µs. The uncertainty for the measurement of the droplet diameter was related to the detection of the droplet perimeter, which is estimated to be ±37 µm. Droplets with diameters smaller than 50 µm were ignored due to the large experimental uncertainty.

We considered typical initial values of parameters for n-dodecane droplets with the water sub-droplets inside them, droplet surface temperatures and ambient gas parameters listed in Table 1. Having substituted these values into Eq. (8) we found the distribution of temperature inside droplets at various time instants. An example of such distributions for a droplet of radius 25 µm, initial homogeneous temperature inside the droplet equal to 434 K and droplet surface temperature equal to 489.47 K (boiling temperature of n-dodecane) is shown in Fig. 2. In this figure, the values of temperature inside droplets are shown as the functions of the distance from the droplet centre (R) normalised by the radius of the water sub-droplet (Rw). For the chosen values of input parameters, for the surface of the droplet R/Rw = 1.88. The curves 1, 2, 3, 4 and 5 refer to time instants 1.1 µs, 11 µs, 0.11 ms, 0.25 ms and 0.5 ms, respectively. As follows from this figure, for curve 5, corresponding to time instant 0.5 ms, the predicted temperature at R = Rw is very close to 373.15 K, which is the boiling temperature of water. We anticipate that at this time instant evaporation of water will lead to puffing above the surface of the water sub-droplet eventually leading to micro-explosion. This time instant is commonly known as time to puffing. We assume that the micro-explosion takes place shortly after the start of puffing.

Plots similar to those presented in Fig. 2 were prepared for all droplet radii, and initial and surface droplet temperatures shown in Table 1. For all these plots, the values of time to puffing were obtained. The values of these times are shown in Fig. 3 as functions of droplet diameters for various initial and surface droplet temperatures. In the same figure the experimentally observed times to puffing for various droplet diameters are shown.

As can be seen from Fig. 3, the values of predicted times to puffing for droplet surface temperature equal to the boiling temperature of n-dodecane are always less than those observed experimentally. In most cases, the orders of magnitude of the predicted times to puffing for other values of droplet surface temperature are the same as those observed experimentally. This does not apply to the results for small droplets with diameters equal to 50 µm for which only one experimental result is available. The predicted increase in times to puffing with the increase in the initial droplet diameters is confirmed by experimental observations, despite considerable scatter of experimental data. The model, however, cannot explain noticeably shorter times to puffing for smaller droplets (with diameters less than 100 µm) than observed experimentally. This could be partly explained by the fact that we linked the start of puffing with the time instant when the water sub-droplet temperature reached the boiling rather than nucleation temperature.

The decrease in the surface temperature leads to an increase in the time to puffing, as expected, although this result cannot be confirmed experimentally at this stage. These results allow us to conclude that the suggested new model, despite its simplicity, is

Table 1  Droplet and gas properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parent droplet radii (Rd) [µm]</td>
<td>25; 50; 100</td>
</tr>
<tr>
<td>Droplet initial composition [vol]</td>
<td>0.15 water + 0.85 n-dodecane</td>
</tr>
<tr>
<td>n-dodecane density (ρf) [kg/m³]</td>
<td>825</td>
</tr>
<tr>
<td>Gas composition</td>
<td>air</td>
</tr>
<tr>
<td>Droplet surface temperatures (Tf) [K]</td>
<td>489.47 (boiling temperature), 470, 450</td>
</tr>
<tr>
<td>Initial droplet temperature (Tdf) [K]</td>
<td>343 and 363</td>
</tr>
<tr>
<td>Gas (air) pressure [MPa]</td>
<td>0.1</td>
</tr>
</tbody>
</table>
able to describe the processes leading to puffing and micro-explosions, not only qualitatively but also quantitatively at least in some cases.

5. Conclusions

A new model for puffing/micro-explosions in water-fuel emulsion droplets is suggested. The model is based on the assumption that a spherical water sub-droplet is located in the centre of a stationary fuel droplet. The fuel is approximated by n-dodecane, the droplet surface temperature is assumed to be fixed, and fuel evaporation is ignored. The penetration of heat from the surface of the droplet to its interior is described by a one-dimensional transient heat conduction equation. This equation is solved analytically inside the composite droplet with the Dirichlet boundary condition at its surface, and the solution leads to a time-dependent distribution of temperature inside the droplet. If the droplet surface temperature is greater than the boiling temperature of water, at a certain time instant the temperature at the interface between water and fuel is expected to reach this boiling temperature. This time instant is referred to as the time to puffing, or micro-explosion delay time.

It is shown that the values of predicted times to puffing for droplet surface temperature equal to the boiling temperature of n-dodecane are always less than those observed experimentally, as expected. The predicted increase in these times with an increase in the initial droplet diameters is also confirmed by experimental observations. These results allow us to conclude that, despite its simplicity, the suggested model is able to describe the processes leading to puffing and micro-explosions not only qualitatively but also in some cases quantitatively.

The models which we developed can be easily generalised to modelling puffing/micro-explosion processes in other composite droplets. Moreover, the equations which we derived, could be applied to the analysis of the processes leading to puffing/micro-explosion in the presence of thermal radiation.

Conflict of interests

None.

Acknowledgements

The current work was carried out as a collaboration between The University of Brighton, supported by the EPSRC, UK (Grants EP/M002608/1 and EP/R012024/1), and the Centre for Automotive Research and Electric Mobility (CAREM), the Universiti Teknologi PETRONAS (UTP), supported by the ministry of higher education Fundamental Research Grant Scheme (FRGS) (Grant FRGS/1/2017/TK10/UTP/01/2) and UTP Graduate Assistant (GA) studentship.

Appendix A. Derivation of Formula (8)

Introducing a new variable

\[ u = (T - T_i) \]

we can simplify Eq. (1) and initial and boundary conditions (3) and (4) to:

\[
\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial R^2} + R P(t, R),
\]

\[
u \bigg|_{t=0} = -T_0 R \quad \text{when} \quad 0 \leq R \leq R_d,
\]

\[
u \bigg|_{R=R_d} = u |_{R=R_d}, \quad k_w \left[ R_w u_k - u \right] |_{R=R_d} = k_f \left[ R_w u_k - u \right] |_{R=R_d}, \quad u \bigg|_{R=R_d} = 0.
\]

where \( T_0 \equiv T_i - T_w(R) \) does not depend on \( R \) in our case. The temperature at the surface of the droplet \( (T_i) \) is assumed to be constant.

Conditions (A1.3) need to be amended by the boundary condition at \( R = 0 \). Since \( T - T_i \) is finite at \( R = 0 \) then \( u \bigg|_{R=0} = 0 \). We look for the solution to Eq. (A1.1) in the form:

\[
u = \sum_{n=1}^{\infty} \Theta_n(t) \nu_n(R),
\]

where functions \( \nu_n(R) \) form the full set of non-trivial solutions to the eigenvalue problem:

\[
\frac{d^2 \nu}{dR^2} + \lambda \frac{\nu}{R} = 0
\]

subject to boundary conditions:

\[
u \bigg|_{R=0} = \nu \bigg|_{R=R_d} = 0
\]

\[
u \bigg|_{R=R_d} = k_w \left[ R_w u_k - u \right] \bigg|_{R=R_d}
\]

\[
\lambda = \frac{k_w}{R_w} R_d^2 \frac{d^2 \nu}{dR^2} \bigg|_{R=R_d} = \lambda_0(R_d)
\]

where

\[
a = \frac{1}{\sqrt{K}} = \sqrt{\frac{k_w}{R_w} R_d^2}
\]

Note that \( j \) has dimension \( 1/\text{time} \). We look for the solution to Eq. (A1.5) in the form:

\[
u(R) = \begin{cases} A \sin(\lambda a_w R) & \text{when} \quad R \leq R_w \\ B \sin(\lambda a_t (R - R_d)) & \text{when} \quad R_w < R \leq R_d. \end{cases}
\]

Function (A1.8) satisfies boundary conditions (A1.6) at \( R = 0 \). Having substituted function (A1.8) into boundary conditions (A1.6) at \( R = R_w \) we obtain:

\[
A \sin(\lambda a_w R_w) = B \sin(\lambda a_t (R_w - R_d)).
\]

\[
A k_w \left[ R_w \lambda a_w \cos(\lambda a_w R_w) - \sin(\lambda a_w R_w) \right] = B k_f \left[ R_w \lambda a_t \cos(\lambda a_t (R_w - R_d)) - \sin(\lambda a_t (R_w - R_d)) \right].
\]

Condition (A1.9) is satisfied when:

\[
A = \pm \left[ \sin(\lambda a_w R_w) \right]^{-1},
\]

\[
B = \pm \left[ \sin(\lambda a_t (R_w - R_d)) \right]^{-1}.
\]

The sign in Eq. (A1.11) can be chosen to ensure that \( u \) in Eq. (A1.4) is negative. Note that we need to choose either ‘+’ or ‘−’ in both expressions.

Having substituted Eq. (A1.11) into (A1.10) we obtain:

\[
k_w \left[ R_w \lambda a_w \cot(\lambda a_w R_w) - 1 \right] = k_f \left[ R_w \lambda a_t \cot(\lambda a_t (R_w - R_d)) - 1 \right].
\]

\[
A k_w \left[ R_w \lambda a_w \cot(\lambda a_w R_w) - 1 \right] = \sqrt{k_w c_w \rho_w} \cot(\lambda a_w R_w) - \sqrt{k_f c_f \rho_f} \cot(\lambda a_t (R_w - R_d)).
\]

\[
k_w - k_f = \frac{k_w}{R_w} R_d^2 \lambda.
\]

Eq. (A1.13) allows us to find a countable set of positive eigenvalues \( \lambda_n \) which can be arranged in ascending order \( 0 < \lambda_1 < \lambda_2 < \ldots \). Note that the negative solutions \( -\lambda_n \) also satisfy Eq. (A1.13) as both sides of this equation are odd functions of \( \lambda \). However, does not satisfy this equation. Having substituted these values of \( \lambda_n \) into Eq. (A1.8) and remembering Eq. (A1.11) we obtain the expressions for eigenfunctions \( \nu_n \) in the form:
When deriving Eq. (A1.15) we took into account Eq. (A1.13). Since all functions $v_{h}(R)$ satisfy boundary conditions (A1.6), function $u$ defined by Expression (A1.4) satisfies boundary conditions (A1.3). Let us expand $R P(t, R)$ in a series over $v_{h}$:

$$
RP(t, R) = \sum_{n=1}^{\infty} p_{n}(t) v_{n}(R),
$$

(A1.16)

where:

$$
p_{n}(t) = \frac{1}{\|v_{n}\|^{2}} \int_{0}^{R} R P(t, R) v_{n}(R) \mathrm{d}R.
$$

Having substituted Eqs. (A1.4) and (A1.16) into Eq. (A.1.1), we obtain:

$$
\sum_{n=1}^{\infty} \Theta_{h}(t) v_{n}(R) = - \sum_{n=1}^{\infty} \Theta_{h}(t) \lambda_{n} v_{n}(R) + \sum_{n=1}^{\infty} p_{n}(t) v_{n}(R).
$$

(A1.17)

When deriving Eq. (A1.17) we took into account that functions $v_{h}(R)$ are orthogonal with weight $b$:

$$
b = \begin{cases} 
    k_{n} a_{n}^{2} = c_{w} \rho_{w} & \text{when } R \leq R_{w} \\
    k_{n} a_{n}^{2} = c_{f} \rho_{f} & \text{when } R_{w} < R \leq R_{d}.
\end{cases}
$$

This means that:

$$
\int_{0}^{R_{w}} v_{n}(R) v_{m}(R) \mathrm{d}R = \delta_{m,n} \|v_{n}\|^{2},
$$

where

$$
\delta_{m,n} = \begin{cases} 
    1 & \text{when } n = m \\
    0 & \text{when } n \neq m.
\end{cases}
$$

See [29] for the discussion of completeness of this set of functions. The norm of $v_{n}$ with weight $b$ is calculated as:

$$
\|v_{n}\|^{2} = \int_{0}^{R_{w}} v_{n}^{2} b \mathrm{d}R
$$

$$
= \int_{0}^{R_{w}} \frac{\sin(\lambda_{n} a_{R})^{2}}{\sin(\lambda_{n} a_{R})} c_{w} \rho_{w} \mathrm{d}R
$$

$$
+ \int_{R_{w}}^{R_{d}} \frac{\sin(\lambda_{n} a_{R} (R - R_{w}))^{2}}{\sin(\lambda_{n} a_{R}) (R_{d} - R_{w})} c_{f} \rho_{f} \mathrm{d}R
$$

$$
= \frac{c_{w} \rho_{w}}{2 \sin^{2}(\lambda_{n} a_{R})} \frac{1}{R_{w} - \sin(2\lambda_{n} a_{R}) (R_{d} - R_{w})}
$$

$$
+ \frac{c_{f} \rho_{f}}{2 \sin^{2}(\lambda_{n} a_{R})} \frac{1}{R_{d} - R_{w}}
$$

$$
= \frac{c_{w} \rho_{w} a_{R}}{2 \sin^{2}(\lambda_{n} a_{R})} + \frac{c_{f} \rho_{f} (R - R_{w})}{2 \sin^{2}(\lambda_{n} a_{R})}
$$

$$
= \frac{k_{n} a_{n}^{2}}{2 R_{d}^{2} a_{n}^{2}}.
$$

(A1.15)

References


