Improving children's perseverance in mathematical reasoning: Creating conditions for productive interplay between cognition and affect

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This paper reports on a small-scale intervention that explored perseverance in mathematical reasoning in children aged 10–11 in an English primary school. The intervention facilitated children's provisional use of representations during mathematical reasoning activities. The findings suggest improved perseverance because of the effect the intervention seemed to have on the bidirectional interplay between affect and cognition. This initially created affectively enabling conditions that impacted on cognition and then created cognitively enabling conditions that impacted on affect. A tentative framework describing this interaction is proposed.

Keywords: Perseverance, mathematical reasoning, affect, cognition, provisional.

INTRODUCTION AND THEORETICAL BACKGROUND

The development of mathematical reasoning is not straightforward; reasoning processes can trace a "zig-zag" route (Lakatos, 1976, p. 42) which necessitates perseverance to navigate cognitive and affective difficulties. The cognitive processes relating to mathematical reasoning have been well documented over the last seventy years (for example, Polya, 1945) and in more recent decades there have been significant theoretical developments in the interpretation of the affective domain in relation to learning mathematics (for example, Hannula, 2011). However, pedagogies to develop children's mathematical perseverance are not yet articulated in the literature. This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The significant interplay between cognitive and affective domains during mathematical learning has been noted at previous CERMEs (Di Martino & Zan, 2013; Hannula, 2011) and this interplay provided the framework for analysing and interpreting the findings in this study.

The importance of reasoning

The central importance of reasoning in mathematics education has been widely argued. For example, Yankelewitz and colleagues (2010) assert that reasoning is crucial in the formulation and justification of convincing mathematical argument. Ball and Bass (2003, p. 28) make a connection between reasoning and the development of mathematical understanding, arguing that in the absence of reasoning, "mathematical understanding is meaningless". They further argue that reasoning has a significant role in the recall of procedures and facts as it is the ability to reason, and not memory that enables a child to reconstruct knowledge when needed. The capacity to reason is therefore a significant factor in children's learning of mathematics and there is value in framing a study with reasoning as its focus.

Mathematical reasoning can be considered to include deductive approaches that lead to formal mathematical proofs and inductive approaches that facilitate the development of knowledge; Polya (1959) broadly interprets these two types of reasoning as demonstrative and plausible reasoning respectively. In this study, my interpretation of mathematical reasoning was based on Polya's (1959, p. 7–9) "plausible reasoning" and includes the use of processes detailed by Mason et al (2010) such as: random or systematic specialising by creating examples; noticing patterns to formulate and test conjectures; generalising and convincing.

Perseverance in reasoning

In this study, I have interpreted perseverance in accordance with common dictionary definitions to mean "persistence in [mathematical reasoning] despite difficulty or delay in achieving success" (OxfordDictionaries, 2014). Lee and Johnston-Wilder (2011, p. 1190) identify perseverance as one aspect of the construct mathematical resilience and argue that it is needed to overcome "mathematical difficulties". Such difficulties arise from the "zig-zag" route that mathematical reasoning typically traces (Lakatos, 1976, p. 42) and can be cognitive or affective in nature.

Overcoming cognitive difficulties necessitates the use of meta-cognitive self-regulatory approaches. For Mason, Burton and Stacey (2010), this is characterised by developing internal monitoring to facilitate deliberate reflection on reasoning processes and their outcomes. Such monitoring might result, for example, in changes in approach or use of representation, or rejection of ideas. This fosters a fallibilistic approach (Charalampous & Rowland, 2013; Lakatos, 1976) to engaging with mathematics and mathematical uncertainty. Mason, Burton and Stacey (2010) emphasise the value of considering three phases of work when engaged in activities involving mathematical reasoning: entry, attack and review. The entry phase, characterised by the making of random trials, and the back and forth movement between phases, exemplifies and facilitates a fallibilistic, self-regulatory approach to mathematical engagement.

Navigating Lakatos' (1976, p. 42) zig-zag path also has affective impact and this necessitates affective self-regulatory responses. Goldin (2000) proposes that affective pathways, comprising rapidly changing emotional states, arise during mathematical problem solving. Malmivuori (2006, p. 152) argues that these emotion responses "direct or disturb" mathematical thinking and activate either active or automatic self-regulatory processes. During active regulation of affective responses, an individual consciously monitors affective responses to inform cognitive decision making. By contrast, automatic affective regulation describes self-regulatory processes that act at a sub-conscious level in which negative emotions can act to impede the higher order cognition involved in reasoning.

Successful engagement with mathematical reasoning can be rewarding and impact on an individual's sense of self-worth. Debellis and Goldin (2006, p. 132) describe mathematical intimacy as an affective structure which portrays an individual's potential "deep emotional engagement" with mathematics. They argue that intimate mathematical experiences can give rise to emotions such as deep satisfaction that impact on self-worth. However, positive mathematical intimacy could be jeopardised by experiencing failure. Debellis and Goldin (2006, p. 138) reason that coping with swings in mathematical intimacy is a "meta-affective capability", the development of which characterises successful problem solvers; this is a further presentation of the perseverance needed to be able to reason mathematically.

THE STUDY

In this study, I sought to improve children's perseverance in mathematical reasoning by applying an intervention that provided children with opportunities to use mathematical representations in a provisional way.

The importance of representation in mathematics learning has been extensively documented and this study draws significantly on Bruner's (1966) modes of representation and Dienes' (1964) Dynamic Principle. However, the notion of provisionality is less widely interpreted within mathematics education.

Provisionality is an idea that is drawn on in information technology (IT) education (Leask & Meadows, 2000). The provisional nature of many software applications enables users to evaluate and refine a product as it is being created. Papert (1980) utilised the provisional nature of programming in designing the LOGO environment. LOGO enables a child to create instructions to move a turtle dynamically on the screen. It facilitates children to conjecture, make trials and use the resulting data to make improvements. Hence, this software enables children to construct understanding through a trial and improvement, conjectural approach to mathematics; the intervention in this study sought to impact on children's cognitive responses by applying a similarly provisional approach to children's use of mathematical representations.

Papert (1980) also notes how the provisional nature of programming impacts on the affective domain. It fosters an attitude that mathematical thinking is fallible (Charalampous & Rowland, 2013), that it concerns trial and improvement and conjecturing rather than the singular pursuit of right or wrong answers. Such an approach, he argues, makes children "less intimidated by a fear of being wrong" (Papert, 1980, p. 23). Hence, by constructing an intervention that enabled children to work provisionally, this study also sought to impact on children's affective responses.

This research took place in an English primary school using an action research approach. The study comprised one Baseline Lesson in which the intervention was not applied, and two Research Lessons in which the teacher applied the intervention to her teaching approach. The teacher selected four children to form the study group based on her assessment that their perseverance in mathematical reasoning was limited and would benefit from improvement. Prior to each of the lessons, the teacher and I selected a mathematical activity that presented opportunities for mathematical reasoning. For the Research Lessons, we discussed how the children could use representations in a provisional way and the teaching strategies that might facilitate this. The teacher then created the detailed plans and taught the lessons.

The fieldwork comprised collecting data from the three lessons, post-lesson interviews with children and an evaluation meeting with the teacher. During the Baseline and Research Lessons, I collected data on the four children relating to the cognitive and affective domains through non-participant observation and by taking photographs of the representations that they made. Audio recordings were made of the children's dialogue during the lessons and I used these to augment the observation notes post-hoc. During observations, I used an approach similar to that used by Schorr and Goldin (2008) in their analysis of filmed lessons to gather data relating to key affective events. For example, I noted the children's manner of engagement, their body position and the speed of their speech. I interviewed the study children immediately after each observation. The focus of the interview was threefold: to check my understanding of what I had observed; to gain the children's interpretation of what had happened and why, and to explore the extent of the children's mathematical reasoning.

This paper reports on the thick data arising from the second Research Lesson pertaining to two of the study group, Lucy and Emily.

FINDINGS AND DISCUSSION

Bidirectional interplay between cognition and affect (Di Martino & Zan, 2013) was evident during Lucy and Emily's mathematical engagement in Research Lesson 2. However, it seemed to operate in different directions at different stages of their thinking. Hence, I have used Mason, Burton and Stacey's (2010) entry and attack phases of problem solving as a temporal framework for the presentation and discussion of findings.

During Research Lesson 2, Lucy and Emily engaged as a pair with the problem:

> A square pond is surrounded by a path that is 1 unit wide. Explore what happens to the path length for different sizes of pond.
>
> Resources available: Cuisenaire rods, pencils, A3 plain paper.

The impact the intervention during the entry phase

During the entry phase (Mason, Burton, & Stacey, 2010), Lucy and Emily used Cuisenaire rods in a provisional way to get a feel for the problem; they explored how the criteria given in the activity could be represented and began to explore how the path size related

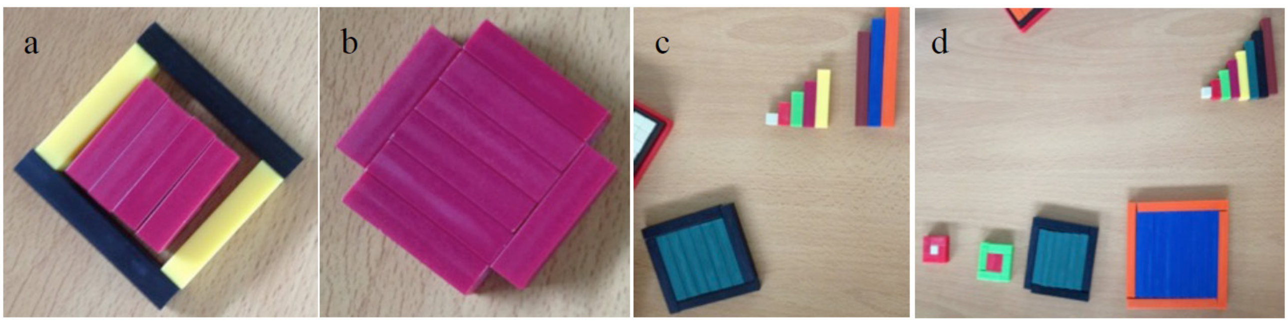

Figure 1: Entry phase trials

to the pond size. In their first three trials (Figure 1a-c) they focused on what it meant for the path to *surround* the pond. They used the information from the first two partially successful trials (Figure 1a-b) to inform their third trial (Figure 1c). This is similar way to in which Papert (1980) described children using the outcomes from their programming in LOGO to fix bugs in code.

The girls' provisional use of representation during the entry phase seemed to impact on their capacity to work with mathematical uncertainty and to adopt a fallibilist approach. Any trials that resulted in failure to meet the criteria set out in the activity, for example those depicted in Figure 1a and 1b did not appear to decrease their engagement or persistence with the activity. Their capacity to work with mathematical uncertainty facilitated their self-regulation and the application of their learning from apparently unsuccessful trials. Emily and Lucy showed no indications of fear, anxiety, bewilderment or reticence that can accompany the beginning of mathematical exploration, when least is known and understood about the problem. Conversely, they seemed highly engaged; they were leaning forwards, constantly exploring the parameters of the problem through their manipulation of the Cuisenaire rods and they alternated between quiet individual construction of examples and paired dialogue to share and develop thinking. The girls portrayed a relaxed appearance during the entry phase; their approach had a sense of playfulness and exploration that could be likened to the unstructured play that Dienes (1964) describes in his Dynamic Principle and this seemed to enable them to experience mathematical uncertainty in a positive way.

During the construction of their third trial, the pair created an ordered arrangement of all ten Cuisenaire rods to serve as a reference of relative lengths and support selection (top right of Figure 1c). In so doing, they noticed that they had selected consecutive rods to create the 6^2 pond and its path. This led them to form the conjecture that began to articulate the relationship between the two dependent variables:

> Lucy: I think it will be if you use 1 [for the pond] then it will be 2 [for the path], if you use 2 then it's going to be 3, so it's [the path] going to be 1 higher than your square number

By the end of the entry phase they had constructed and ordered four examples (Figure 1d). They appeared to create each example by randomly selecting a Cuisenaire rod and using this as the basis to create one example; this use of random specialisation typifies the entry phase trials (Mason, Burton, & Stacey, 2010). This facilitated cognitive developments that enabled the girls to notice and formulate conjectures about the emerging patterns between the width of the pond and side length of path and to begin to articulate this relationship.

Hence, during the entry phase, the provisional way in which the girls used representations seemed to foster the emergence of affectively enabling responses and this enabled cognitive developments in mathematical reasoning. The impact of the girls' provisional use of representation during the entry phase is depicted in Figure 2.

The impact of the intervention in the attack phase

The transition to the attack phase was indicated by the girls' use of systematic specialisation (Mason, Burton, & Stacey, 2010). Having organised the data generated through random specialisation into an ordered sequence (Figure 1d), the girls then used the provisional nature of their representations to create gaps between the examples, apparently to identify and accommodate missing data. They then represented all the ponds in an ordered sequence from 1^2 to 9^2 using Cuisenaire rods (Figure 3).

The girls then switched to a more permanent representation in the form of a table (Figure 4). This representation does not simply illustrate total amounts relating to pond size and path lengths. Rather, it includes significant detail relating to the mathematical structures that underpin the relationship between the dependent variables of pond size and path length. Each example of the pond described its width squared, its total value and the odd/even property of this to-

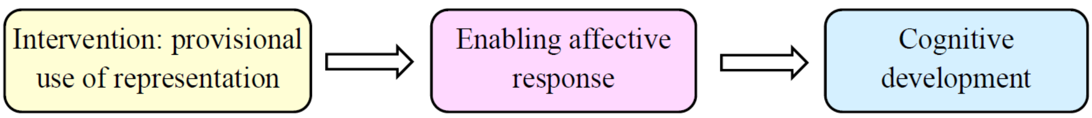

Figure 2: Impact of the intervention during the entry phase

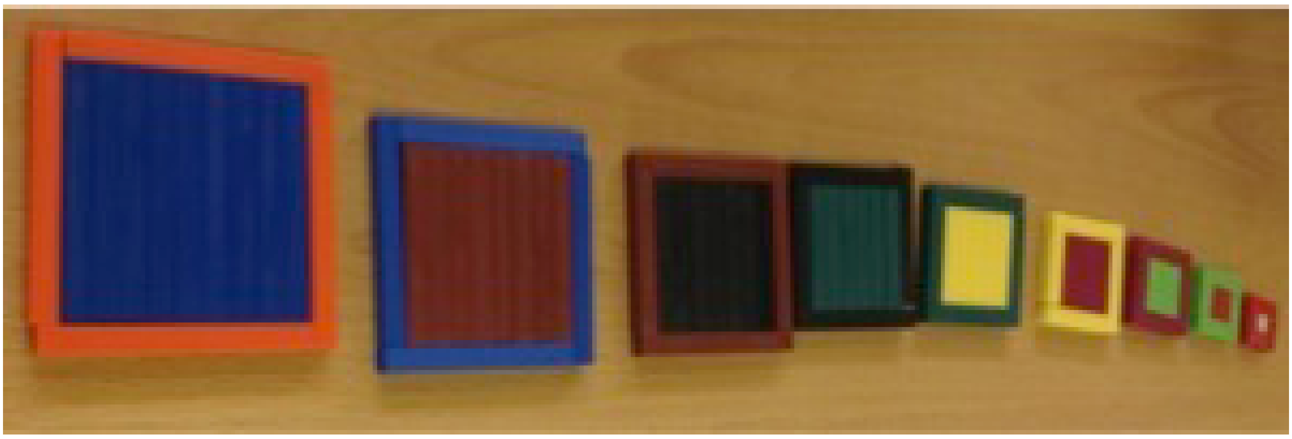

Figure 3: Systematic representation of ponds with widths 1–9

tal. Each example of the path is similarly described by side length multiplied by 4, the total value of the path length and the even nature of these totals. The girls also noted that each total was a multiple of 4. Interestingly, they realised that their recording had not been totally consistent in representing the x4 aspect of the path side length and this led them to underline the x4 component. Whilst there was no evidence in this lesson that the girls became overtly stuck, and hence no necessity to overcome this, they did persevere in formulating and articulating the reasoning for the patterns they observed. Emily's original response to the challenge of explaining the patterns they had identified resulted in a sentence that she was initially unable to complete:

Emily: All the paths are in the four times table. They have to be in the four times table because...

The girls persisted and utilised their understanding of the structures they had identified to formulate their reasoning for the observable patterns. This is captured on the right of Figure 4. In the post-lesson interview, the girls re-visited this:

13 Emily: We noticed about the path, because there's 4 sides to the path, we need 4 sides of the path, so you need to times it by whatever number the length of the path is. So then it's the 4 times table because there

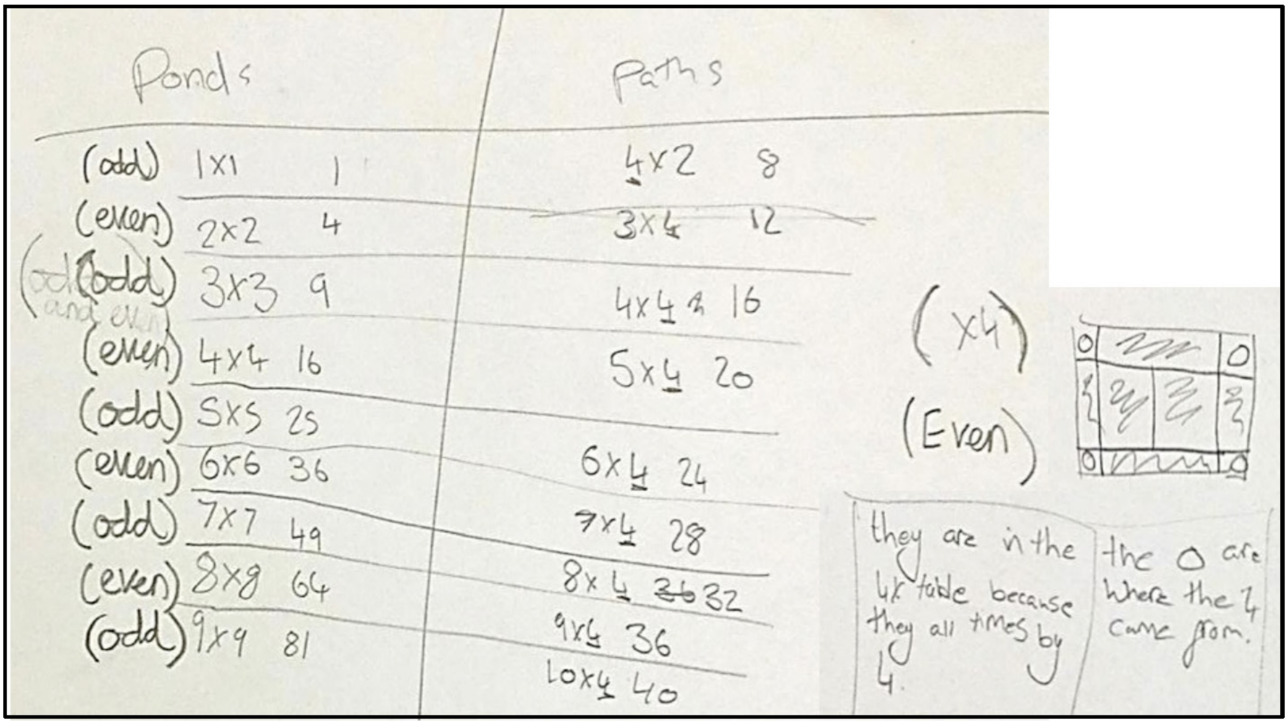

Figure 4: Lucy and Emily's table of findings

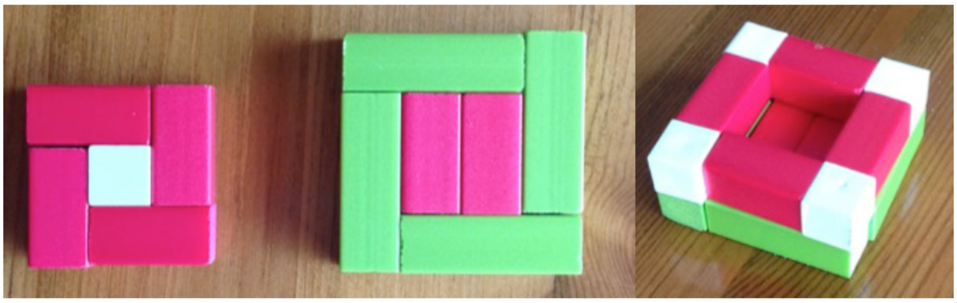

Figure 5: Representations created to support reasoning in line 69

are 4 sides and all of them, the numbers are even because they are all in the 4 times table

69 Lucy: Because it expands so you need to add 4 each time you go up

The diagram on the right of Figure 4 supports the reasoning expressed in line 69. In the interview, the girls re-created this image using Cuisenaire rods; Figure 5 shows how the path surrounding the 12 pond is positioned on top of the path surrounding the 22 pond with the gaps at each corner filled by four rods, each of length 1. There are similarities between the representations drawn in Figure 4 and constructed in Figure 5 and the girls' second trial (Figure 1b); the initial provisional explorations using the Cuisenaire rods, and in particular the example in Figure 1b seems to have helped the girls to understand the structures underpinning the growth of the path size. This understanding enabled Lucy to articulate the reasoning in line 69. The depth of understanding and the extent of the reasoning that the girls achieved resulted in positive affective responses. As in the entry phase, both girls remained highly engaged in the activity throughout the attack phase and took every opportunity presented to talk with the teacher about their findings and seemed eager to share the reasoning that they were constructing.

In the evaluation meeting following the Research Lesson, the teacher reported the impact of the girls' provisional use of representations during the attack phase on their cognitive and affective domains:

18 Teacher: I think [the provisional use of representation] helped them explain their reasoning more and therefore that helped them sustain their interest because they could explain more, because they had something to work from, to explain with. Their level of reasoning was amazing.

96 Teacher: [Lucy's] very proud of the work she's done [in the project]. I only have to mention it and a smile spreads across her face.

108 Teacher: I have seen some improvement in [Emily's] perseverance and resilience [...] in the past she would very much continue to follow a path even though it was wrong [...]. She's been able to stop mid way and realise it's wrong and have to go back to the beginning.

In line 18, the teacher exclaims about the level of the girls reasoning. In the baseline lesson, the girls were able to notice and articulate patterns, but not reason about why these occurred, hence there was a significant contrast with the extent and depth of their reasoning between the baseline lesson and the second research lesson.

The teacher also makes two connections in line 18. First, she makes a link between the girls' provisional use of representation and their articulation of mathematical reasoning. Second, she perceives that the positive cognitive developments contributed to the girls' sustained engagement and curiosity. The impact on Lucy's affective domain appeared to continue beyond the Research Lesson. Lucy's apparent sense of pride (line 96), suggests that she may have experienced developments in mathematical intimacy; that she was emotionally engaged and achieved a sense of satisfaction and self-worth through her cognitive mathematical activity (DeBellis & Goldin, 2006). Line 108 suggests that Emily may have increased her capacity to actively self-regulate (Malmivuori, 2006); this perhaps arises from developments in her capacity to work with mathematical uncertainty which may have arisen through working in a provisional way.

It appears that the provisional use of representations in the attack phase impacts first on the cognitive domain and second on the affective domain; a reversal

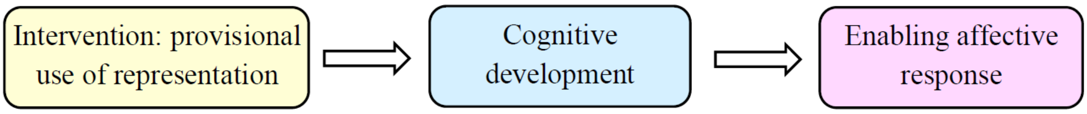

Figure 6: Impact of the intervention during the attack phase

of the processes emerging in the entry phase. This relationship is depicted in Figure 6.

CONCLUSION AND NEXT STEPS

This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The girls' provisional use of Cuisenaire rods appeared to have an enabling affective impact during the entry phase. This facilitated cognitive developments in reasoning as it supported them to behave in an exploratory way, to make and learn from trials, work with mathematical uncertainty and begin to formulate conjectures. In the attack phase, their provisional use of representation seemed to enable the girls to develop systematic approaches to their creation and organisation of trials. This led to their noticing patterns, understanding the underpinning mathematical structures, and using this to persevere in formulating reasoning. It seems that positive bidirectional interplay (Di Martino & Zan, 2013) between affect and cognition, facilitated by the intervention, resulted in improved perseverance in mathematical reasoning. A tentative analytic framework detailing these interactions and synthesising Figures 2 and 6, is depicted in Figure 7.

In the next phase of this research, I plan to work with two classes of children aged 10–11 in different schools to further test the impact of the intervention on children's perseverance in mathematical reasoning.

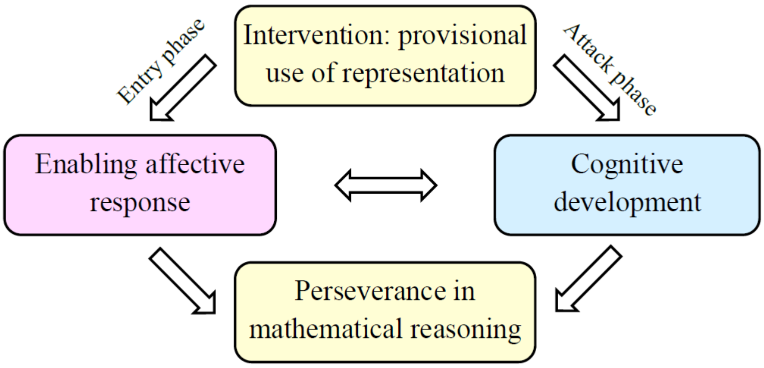

Figure 7: Tentative analytic framework

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