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What Makes an Effective Representation of Information: A Formal Account of Observational Advantages

Gem Stapleton · Mateja Jamnik · Atsushi Shimojima

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Abstract In order to effectively communicate information, the choice of representation is important. Ideally, a chosen representation will aid readers in making desired inferences. In this paper, we develop the theory of *observation*: what it means for one statement to be observable from another. Using observability, we give a formal characterization of the *observational advantages* of one representation of information over another. By considering observational advantages, people will be able to make better informed choices of representations of information. To demonstrate the benefit of observation and observational advantages, we apply these concepts to set theory and Euler diagrams. In particular, we can show that Euler diagrams have significant observational advantages over set theory. This formally justifies Larkin and Simon's claim that "a diagram is (sometimes) worth ten thousand words".

1 Introduction

When we want to share and understand information, we need to represent it in some notation. There is a plethora of notations available to us for this purpose. Even when a notation has been selected, we must still choose how to

G. Stapleton
University of Brighton
Brighton, UK
E-mail: g.e.stapleton@brighton.ac.uk

M. Jamnik
University of Cambridge
Cambridge, UK
E-mail: mateja.jamnik@cl.cam.ac.uk

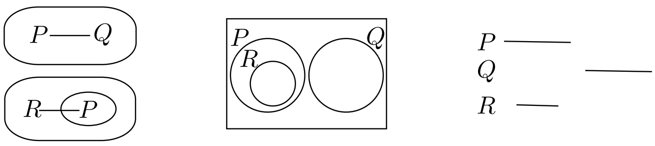
A. Shimojima
Doshisha University
Kyoto, Japan
E-mail: ashimoji@mail.doshisha.ac.jp

use its syntax to represent the desired information. This paper is concerned with the relative advantages of one choice of representation of information over another. Many factors can contribute to such advantages. For instance, graphical features, such as the way in which colour is used, and visual clutter (or lack thereof) can impact on the ease with which information can be extracted from a representation. The particular focus of this paper is on what we call *observational advantages*; this term, and others, illustrated informally in the early sections of this paper, will be formalized at appropriate points as we proceed.

As a simple example, suppose we wish to represent these two facts about three sets, P, Q and R: nothing is in both P and Q, and everything in R is also in P. There are many notations that can express this information: five examples are illustrated in Figure 1. The first two are sentential and the

$$(P \cap Q) = \emptyset \qquad (P \sqcap Q) \sqsubseteq \bot$$
$$R \subseteq P \qquad R \sqsubseteq P$$

(1) Set Theory (2) Description Logic



(3) Existential Graph (4) Euler Diagram (5) Linear Diagram

Fig. 1 Multiple choices of representation.

last three are diagrammatic. There is little syntactic difference between (1), the set theory representation, and (2), the description logic representation [1]. Both utilize two statements[1] to express the desired information. Each of these statements has a single *meaning-carrying relationship*. By meaning-carrying relationship we mean a relation on the syntax of the statement that carries semantics and evaluates to either 'true' or 'false'. Thus, the idea of meaning-carrying relationship is similar to what Shin referred to as 'representing fact' in her seminal work on the systems of Venn diagrams [20]. It plays an important role in both our work as well as hers. The first statements in (1) and (2) assert that the intersection of the two sets is empty. The meaning-carrying relationship in (1), and similarly for (2), is that $(P \cap Q)$ and $\emptyset$ are written either side of $=$. The statement $(P \cap Q) = \emptyset$ evaluates to either true or false, depending on the interpretation of P and Q as sets. The second statements in (1) and (2) describe a subset relationship; here, the meaning carrier in (1) is that R is written on the left of $\subseteq$ and P is written on the right.

[1] By statement, we mean a syntactic entity (in any representation system) that represents some information. For example, a set-theoretic sentence is a single statement, and so is an Euler diagram.

Representation (3) is an existential graph [6, 15], comprising of two components. The top-most component expresses the disjointness of P and Q. It is directly read as 'it is not the case that something is in both P and Q', since the closed curve, called a cut, represents negation and the line represents the existence of an element in the sets denoted by its end points. The second component reads 'it is not the case that something is in R and not in P.' Like the two sentential representations, each component of the existential graph has a single meaning-carrying relationship. The top component's meaning carrier is the relationship that P–Q is enclosed by the cut. The bottom component has a similar meaning carrier: the inner part is enclosed by the outer-most cut.

The other two representations of this information each comprise a single statement which has *many* meaning-carrying relationships. Representation (4), the Euler diagram, uses non-overlapping curves to express the disjointness of P and Q and, similarly, curve containment to assert that R is a subset of P; note that our use of Euler diagrams does not assume any existential import – present regions can represent empty sets. Here, two meaning-carrying relationships (namely, disjointness and containment) are exploited to convey the desired information. As a consequence of the way in which Euler diagrams are formed, *additional* meaning-carrying relationships are evident. Most obviously, the non-overlapping relationship between Q and R is a meaning carrier. Thus, from the Euler diagram we can *observe* the statement that Q and R are disjoint. By contrast, this statement cannot be observed from (1), (2) or (3) but must be *inferred* from the statements given. This is an example of an *observational advantage* of the Euler diagram over the previous three representations of this information. The linear diagram [5], used in representation (5), has similar advantages. This diagram represents the sets using lines. One line completely overlapping with another represents a subset relationship and lines with no overlap represent disjoint sets. Again, we can see that through representing the desired information, the linear diagram also allows us to observe that Q and R are disjoint.

Thus, among the five notations in Figure 1, three allow only one meaning-carrying relationship per statement, while two allow multiple meaning-carrying relationships per statement. This difference in the design of notations makes a crucial difference to what can and cannot be observed from a statement, and thus to the cognitive values of notational systems. We will return to this point later when we discuss observational properties of notations in a more systematic manner. Here, we only note that this difference between single and multiple meaning carriers should not suggest any sharp dichotomy of sentential and diagrammatic notations, as is clear from the example of existential graphs that allow only one meaning carrier per statement.

Representations of information that allow statements to be observed as true, without the need for inference, can be considered advantageous. In this paper, through developing the theory of observation and observational advantages, we provide a framework in which relative benefits of one representation of information over another can be characterized and understood. Our formal account will lead to more informed choices when representing information: we will be able to advantageously exploit the use of meaning-carrying relationships when considering the information that end-users may wish to derive.

We demonstrate how to apply our theory of observation and observational advantages to set theory and Euler diagrams. These notations have been selected for a number of reasons. First, Euler diagrams form the basis of many visualization techniques that have been employed for representing information, such as Bubble Sets [4], EulerView [22,21], iCircles [23], and KelpFusion [14]. They also form the basis of a number of diagrammatic logics, such as Euler/Venn [26], spider diagrams [10], and constraint diagrams [11]. Their prominent role in both information visualization and diagrammatic reasoning makes our case study particularly relevant to these areas. Moreover, set theory is widely understood and is a contrasting notation to Euler diagrams. Its syntactically similar nature to description logic, used for ontology engineering, means that many of our arguments about set theory readily adapt to description logic. Lastly, by comparing set theory and Euler diagrams, we are able to show that the latter have numerous observational advantages over the former. That is, we provide a theoretical way of capturing the often seen observational advantages of diagrams over sentential systems.

The paper is structured as follows. We develop the theory of observation in Section 3, including the concepts of observational completeness and observational devoidness. The idea of an observational advantage is presented in Section 4. After this, the theory of observation is developed for set theory and Euler diagrams. Section 5 presents the formalization of these two notations. The circumstances under which information can be observed from set-theoretic sentences and Euler diagrams are given in Section 6. We demonstrate that set-theoretic sentences are *observationally devoid* in Section 7, where we also establish that Euler diagrams are *observationally complete*. As a consequence, Euler diagrams have many observational advantages over set-theoretic sentences, shown in Section 8. We conclude and discuss future directions in Section 9. Proofs for all of the lemmas and theorems can be found in the supplementary material.

2 Background

Our research is concerned with producing a theory of observation in order to formalize the idea of an observational advantage. Our motivation is to allow more informed choices to be made when choosing representations of information. By choosing representations with beneficial observational advantages, we reduce the need for inference and, consequently, proof systems. Prior work has used the idea of observation, and the related idea of free-rides, within the context of defining proof systems.

Indeed, the main tool of our analysis is the concept of observation, which has been formalized as an inference rule in proof systems [2,26]. Roughly, this inference rule lets us focus on one of the pieces of information expressed in a statement and re-express it in another statement. This may seem trivial because such an inference rule, by definition, adds no information to a proof that has not been already expressed in a previous statement. This is particularly true for a proof involving only sentential representations. Many sentences support a unique 'meaning-carrying relationship' and, thus, when the observation rule is applied to such a sentence, it ends up restating its unique meaning in an identical or synonymous sentence. This does not constitute significant progress in a proof. Since proof-theoretic studies have been predominantly done using sentential proofs, the concept of observation has never played an important theoretical role.

The situation is very different in a proof involving diagrammatic representations. While a sentence often has only one meaning-carrying relationship, a diagram typically has many meaning-carrying relationships, as demonstrated in Figure 1 and discussed in Section 1. Thus, the observation rule applied to a diagram can yield many different statements, depending upon which meaning-carrying relationship it operates. This makes observation an interesting rule in the realm of diagrammatic proofs, since the choice of statements observed in this step could change the course of a proof entirely.

For this reason, Barwise and Etchemendy's *Hyperproof* [2] incorporated the observation rule in their hybrid system of proofs involving both pictorial diagrams of a blocks world and first-order sentences. Swoboda and Allwein [26] were more conscious about the difference between the observation rule and other inference rules. Drawing upon Dretske's classification of various cases that are commonly described as "somebody's seeing that something is the case" [7], Swoboda and Allwein called for the distinctive treatment of the observation rule, which consists of visual perception and the restatement of the information thus obtained, from other rules that require observational operations that go beyond perception and restatement. This way, we can investigate the observational advantages of a statement in a proof-theoretic manner, by identifying the set of statements *observable* from it, as opposed to those statements that must be inferred. This approach is particularly important if we are to investigate the observational advantages of visual representations in general, including both diagrammatic and sentential representations.

In fact, this approach is suited to clarify how the 'free ride' capacities of diagrams [17,18] influence their observational advantages. In a diagrammatic system, expressing chosen information in a diagram often results in the expression of other pieces of information that are consequences of the chosen information; informally, these other pieces of information are free rides. We will shortly see an example of such a free ride in the case of Euler diagrams, while many other diagrammatic systems, including systems of maps, line drawings, and geometry diagrams, also have this capacity [3,8,12]. When such free rides occur, the task of inferring consequential information from the given premises is replaced by the task of expressing the premises in diagrams and observing the consequences that are automatically expressed there. Using diagrams in this way lets us obtain consequential information with little inferential effort, giving 'free rides' to the consequences. As is clear from this description, the main force of a free ride consists of the fact that the diagram has a number of consequences *observable* from it, not just *inferable* from it. Thus, an analysis based on the observation rule is essential to formally capture the advantages (and disadvantages) relating to inference brought about by free rides.

Specifically in terms of set theory[2] and Euler diagrams (the focus of the case study later in this paper) it has long been believed that the latter have advantages, relating to inference, over the former. Recent work, such as [16], has empirically established that inference tasks are performed more accurately with Euler (and Venn) diagrams than when using sentential representations. However, there has been no previous attempt at theoretically classifying such advantages of Euler diagrams over set theory. The theory of observation and observational advantages developed in this paper allows us to formally address the adage 'a picture is worth a thousand words', directly related to diagrams in [12], by quantifying rigorously the observational advantages of Euler diagrams over set theory.

3 Observation of Information

Generally, a visual representation, be it a diagram or a written sentence, expresses meaning by having symbols standing in a certain visual or spatial relationship. For example, a metro map expresses the information that Piccadilly Circus is directly connected to Green Park by having a node labelled 'Piccadilly Circus' connected to a node labelled 'Green Park' with a line, as shown in Figure 2. An Euler diagram expresses the information that P is a subset of Q by having the curve labelled P included inside the curve labelled Q. Similarly, the set-theoretic sentence $P \cap R \subseteq Q$ expresses the information that P intersection R is a subset of Q by having labels P, Q, and R standing in a certain spatial relationship mediated by the special symbols $\cap$ and $\subseteq$.

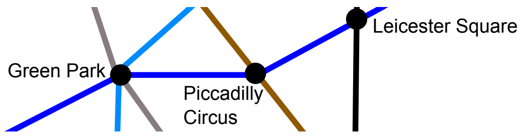


Fig. 2 Observing information from a metro map.

When a statement, σ, expresses a certain meaning by having a visuo-spatial relationship, δ, among its syntactic elements, we call δ a *meaning-carrying relationship* in σ. Note that the term 'meaning' in this definition is confined to what is sometimes called an 'assertive content,' a piece of information asserted by a statement that evaluates to either true or false in an interpretation. For example, the information that Piccadilly Circus is directly connected to Green Park counts as a meaning of the metro map in Figure 2, as it is an assertive

[2] By the term 'set theory' we mean a symbolic representation of sets, such as $P \cap Q$, and the assertion of relationships between sets, such as $P \cap Q \subseteq R$.

content of the metro map in Figure 2 — if it were not true, the map would be considered inaccurate, at least partially. Thus, the visuo-spatial relationship holding in the metro map responsible for the expression of this information counts as a meaning-carrying relationship. By contrast, consider the 'meaning' of the leftmost node, labeled 'Green Park,' in the map. Although the node denotes the Green Park station, its denotation does not count as a meaning in our terminology, since stations are not the kind of things to which truth and falsehood can be attributed and therefore not the kind of things whose truth can be asserted. Accordingly, the visuo-spatial relationship with which that node denotes the Green Park station — that is, the fact that it is a black circle with the label "Green Park" in its spatial proximity — does not count as a meaning-carrying relationship.

Similarly, the denotation of the term $P \cap R$ in the set-theoretic sentence $P \cap R \subseteq Q$ does not count as a meaning, since, again, the intersection of the sets P and R is not the kind of thing to which truth or falsehood can be attributed. Accordingly, the visuo-spatial relationship that enables this term to denote the relevant set — that is, the fact that the labels P and R have the symbol $\cap$ in between — does not count as a meaning-carrying relationship in this set-theoretic sentence. The meaning-carrying relationship in it is rather the fact that the term $P \cap R$ is to the left of the letter Q over the symbol $\subseteq$, since the information carried by this relationship — that is, that P intersection R is a subset of Q — is a piece of information asserted by the set-theoretic sentence that evaluates to either true or false in an interpretation.

What we call *observation* selects some of these meaning-carrying relationships holding in a representation and produces another representation that supports just enough relationships to express the meanings carried by the selected relationships and nothing stronger. Take the Euler diagram d in Figure 3, for example. Observation may select the spatial relationship between the curves labelled P and Q in d and produce d_1. When observation selects the spatial relationship between the curves labelled P and R in d, it produces d_2. Similarly, d_3 is the result of observation operating on the spatial relationship between curves labelled R and Q in d. In each of these cases, observation operates on a 'local' meaning-carrying relationship involving just two of labelled curves and results in semantically weaker diagrams. If it operates on the relationship among all three labelled curves, it results in d itself. Thus, d_1, d_2, d_3, and d itself are all observable from d. By contrast, d_4 is not observable from d, even though it is semantically entailed[3] by d. By including the curve labelled S, d_4 supports extra visuo-spatial relationships that are not necessary to express the meaning that is carried by any visuo-spatial relationships in d.

In some cases, observation produces a set-theoretic sentence out of an Euler diagram. For example, it may operate on the spatial relationship between the curves labelled P and Q in d and produce the set-theoretic sentence $P \cap Q = \emptyset$. Note that syntactic elements P, Q, $=$, and $\emptyset$ stand in a certain spatial rela-

[3] We use the term semantically entailed in the standard, model theoretic way: one set of statements, Σ, semantically entails a single statement, σ, if all the models for Σ are models for σ.

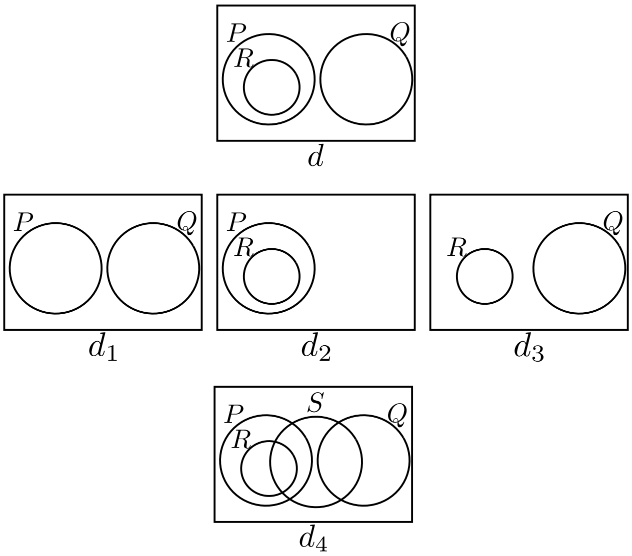


Fig. 3 Observing information from Euler diagrams.

tionship in this set-theoretic sentence, and this relationship is just enough to express the meaning carried by the spatial relationship of the curves labelled P and Q in d. When operating on the spatial relationships between different pairs of curves in d, observation may produce the set-theoretic sentences $R \subseteq P$ and $R \cap Q = \emptyset$. This is analogous to the ways the diagrams d_1, d_2, and d_3 are observed from d.

All of these are examples of observation applied to an Euler diagram to produce another statement, be it a diagram or a set-theoretic sentence. But observation can also be applied to a set-theoretic sentence to produce another set-theoretic sentence or an Euler diagram. Observation works in a similar way in these cases too, which we discuss next.

What is special about set-theoretic sentences, and sentences in many natural or artificial languages, is the fact that a whole sentence, rather than a part of it, often determines its meaning-carrying relationship. Thus, even though label R is connected to label Q with the symbol $\subseteq$ in the set-theoretic sentence $P \cap R \subseteq Q$, this spatial relationship is not a meaning carrier in this sentence. For if it were, we could read off the information that R is a subset of Q from this sentence, which is clearly not valid. Indeed, reinstating the omitted brackets, which should formally be present, the set-theoretic sentence $P \cap R \subseteq Q$ becomes $(P \cap R) \subseteq Q$. This latter presentation more clearly delineates R from the special symbol $\subseteq$, indicating that it is the whole of the left hand side of $\subseteq$ that is asserted to be a subset of Q. By the same token, the same relationship cannot be a meaning carrier in the set-theoretic sentence $P \cup R \subseteq Q$ even though, in this case, the information one would extract happens to be a valid conclusion of what is expressed by the entire sentence.[4] Thus, in the case of set-theoretic sentences, we are simply not supposed to focus our attention on a partial structure of the given sentence and extract information out of it. Rather, we are to parse the entire structure of the sentence and read off the meaning carried by that structure.

This is in contrast to our reading practices for most diagrams. To extract information from a metro map, one does not need to parse its entire structure. Just focusing on a particular pair of nodes and the edge connecting them, one can read off meaning, and allowing such a 'partial' reading is a function of the metro map. Focusing on other pairs of nodes (or sets of nodes), one obtains other meanings. Many meaning-carrying relationships hold in a metro map, and one can focus on an arbitrary relationship to extract information carried by it. As our examples concerning the Euler diagram d in Figure 3 have suggested, an Euler diagram accepts partial reading that focuses on its local relationships. Unlike the case of set-theoretic sentences, meaning-carrying relationships do not have to be determined by the whole Euler diagram. Partial structures in a diagram can be taken to carry meaning.

Although the contrast we have just drawn between set-theoretic sentences and Euler diagrams is applicable to many sentential representations and diagrammatic representations, it does not define a strict dichotomy between two classes of representations. A language consisting of sentences which have very particular structures, such as conjunctive normal forms in first-order logics, does allow one to focus on a particular conjunct in a sentence and extract meaning out of it. In fact, allowing partial reading seems to be a function of such a stylized language. However, partial reading in this case is possible only at the price of severely restricting the syntax of the language. In a more common language where wide-scope negation and disjunction are permitted, for example, we do not read off any assertion from a partial structure of a sentence. Rather, we read a sentence as a whole, to parse its entire construction and correctly recognize the net assertion made by it. The flip side of this situation is that, if one introduces negation or disjunction in a system of Euler diagrams, the system no longer accepts an unconditional partial reading. We may see that the curve labelled P is included in the curve labelled Q in a negated Euler diagram, but the diagram does not thereby assert that P is a subset of Q. Such a diagrammatic system loses the capacity to allow partial reading, and thus becomes, in this respect, more similar to many sentential representation systems. Generally, what counts as a meaning-carrying relation in a representation depends on the entire design of the notational system in question — that is, for which capabilities it has been developed and because of which capabilities it has been used (see [19] for an account of how a specific semantic convention of a notational system arises).

Meaning-carrying relationships are central to our notion of observation. In particular, observation from a single statement, σ, is a binary relationship

[4] From this point forward, we will **not** adopt the standard convention of omitting brackets where no semantic ambiguity arises, in order to avoid potential confusion as to the structures in set-theoretic sentences that are meaning carriers.

between σ and another statement, σ_o, denoted $\sigma \rightsquigarrow \sigma_o$, which ensures the following properties hold:

1. some of the meaning-carrying relationships holding in σ hold in σ_o, and
2. σ_o supports just enough relationships to express the meanings carried by the selected relationships in σ and nothing stronger.

These two properties imply that σ *must semantically entail* σ_o but, as we have seen above, observation is not equivalent to semantic entailment. In practice, any observation relation must be formally defined by appealing to the syntax of the notations in question, and consider how the meaning-carrying relationships arise from the manner in which the semantics are defined.

In terms of a set of statements, Σ, the *only* meaning-carrying relationships in Σ arise within each individual statement in Σ. This implies that the only statements observable from Σ must be observable from one of the elements in Σ. Therefore, we have the following definition, which assumes an observation relation is defined between pairs of statements.

Definition 1 Let Σ be a finite set of statements and σ_o be a single statement. Then σ_o is **observable** from Σ, denoted $\Sigma \rightsquigarrow \sigma_o$, iff σ_o is observable from some statement, σ, in Σ. The set of statements that are observable from Σ is denoted $\mathcal{O}(\Sigma)$.

To conclude this section, we introduce the concept of a set of statements, Σ, being *observationally complete* with respect to some set of statements, $\Sigma_{\models}$. We can think of $\Sigma_{\models}$ as being the set of conclusions we wish to draw from Σ, some or all of which may be observable.

Definition 2 Let Σ and $\Sigma_{\models}$ be finite sets of statements. Then Σ is **observationally complete** with respect to $\Sigma_{\models}$ if

$$\Sigma_{\models} \subseteq \mathcal{O}(\Sigma).$$

Definition 3 Let Σ and $\Sigma_{\models}$ be finite sets of statements. Then Σ is **observationally devoid** with respect to $\Sigma_{\models}$ if

$$\Sigma_{\models} \cap \mathcal{O}(\Sigma) = \emptyset.$$

To illustrate the ideas of observational completeness and devoidness, we appeal to $\mathcal{D} = \{d\}$, where d is in Figure 3. Taking $\mathcal{D} = \Sigma$ and $\Sigma_{\models} = \{d_1, d_2, d_3\}$, we see that $\mathcal{D}$ is observationally complete. However, taking instead $\Sigma_{\models} = \{d_4\}$, $\mathcal{D}$ is observationally devoid.

Observationally complete representations of information can be considered powerful in the following sense: if $\Sigma_{\models}$ contains all the statements whose truth we wish to establish then we can simply observe them all to be true from Σ. Likewise, observationally devoid representations of information are weak, since we cannot observe any statement in $\Sigma_{\models}$ to be true, but must infer them all to be true.

4 Observational Advantages

In this section, we define the new concept of an *observational advantage*, which generalizes the idea of a free ride introduced previously by Shimojima [17, 18]. Our definition of an observational advantage requires three key notions to be defined: *semantic entailment*, *semantic equivalence*, and what it means for a statement to be *observable* from a set of statements. These three key notions must be defined for particular notations (we cannot give notation-independent definitions) and we assume their existence in order to define observational advantages; we give formal definitions for these three notions in the context of set theory and Euler diagrams later in the paper.

The original idea of a free ride assumes a semantics-preserving translation from one notation, N_1, into another notation, N_2, such that the translation ensures the original statements are observable from the resulting statements. We can explain free rides in detail by appealing to our chosen case study: set theory and Euler diagrams. Suppose we have a finite set of set-theoretic sentences, $\mathcal{S}$, where a set-theoretic sentence is a statement that asserts either set equality or a subset relationship. Further, suppose that we then identify a semantically equivalent[5], finite set of Euler diagrams, $\mathcal{D}$, such that each statement, s, in $\mathcal{S}$ is observable from a diagram, d, in $\mathcal{D}$; we can view $\mathcal{D}$ as being a translation of $\mathcal{S}$. Then the set-theoretic sentences that are observable from the diagrams in $\mathcal{D}$ but not from $\mathcal{S}$ are *free rides* from $\mathcal{D}$ given $\mathcal{S}$.

More formally, suppose Σ and $\hat{\Sigma}$ are finite, semantically equivalent sets of statements and let σ be a statement. For σ to be a *free ride* from $\hat{\Sigma}$ given Σ the following must hold:

1. the set Σ contains only statements observable from $\hat{\Sigma}$,
2. σ is not observable from Σ, and
3. σ is observable from $\hat{\Sigma}$.

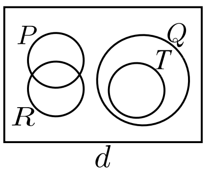


Fig. 4 Free rides.

For example, take

$$\Sigma = \mathcal{S} = \{(P \cap Q) = \emptyset, (R \cap Q) = \emptyset, T \subseteq Q\},$$

which contains three set-theoretic sentences, and $\hat{\Sigma} = \mathcal{D} = \{d\}$, where d is in Figure 4. The free rides from $\mathcal{D}$ given $\mathcal{S}$ are the set-theoretic sentences that

[5] We use the term semantically equivalent in the standard, model theoretic way: two (sets of) statements are semantically equivalent if they have the same models.

one can observe to be true from $\mathcal{D}$ but which need to be inferred, and cannot simply be observed, from $\mathcal{S}$. For instance, we can *observe* both $(P \cap T) = \emptyset$ and $(R \cap T) = \emptyset$ from d but both of these set-theoretic sentences must be *inferred* from $\mathcal{S}$; in the former case, $(P \cap T) = \emptyset$ can be inferred from $(P \cap Q) = \emptyset$ and $T \subseteq Q$. By contrast, whilst the set-theoretic sentence $(P \cap Q) = \emptyset$ can be observed from $\mathcal{D}$ it can also be observed from $\mathcal{S}$, so it is not a free ride from $\mathcal{D}$. Free rides are examples of what we call *observational advantages* of the Euler diagram over the original set theory representation of information. An important difference between observational advantages and free rides is that observational advantages do not require the set Σ to contain only statements observable from $\hat{\Sigma}$.

Definition 4 Let Σ and $\hat{\Sigma}$ be finite, semantically equivalent sets of statements. Let σ be a statement. If

1. σ is not observable from Σ, and
2. σ is observable from $\hat{\Sigma}$

then σ is an **observational advantage** of $\hat{\Sigma}$ given Σ. The set of all observational advantages of $\hat{\Sigma}$ given Σ is denoted $\mathcal{OA}(\hat{\Sigma}, \Sigma)$.

The abstract nature of Definition 4 means that it readily applies to any (not necessarily distinct) notations over which statements are formed, provided the aforementioned three key notions are defined. Furthermore, an observational advantage σ can be expressed in either the representation of Σ or $\hat{\Sigma}$.

Lemma 1 *Let Σ and $\hat{\Sigma}$ be finite, semantically equivalent sets of statements. Let σ be an observational advantage of $\hat{\Sigma}$ given Σ. Then Σ semantically entails σ.*

Taking $\Sigma = \mathcal{S} = \{(P \cap Q) = \emptyset, (R \cap Q) = \emptyset, S \subseteq Q\}$, and $\hat{\Sigma} = \mathcal{D} = \{d\}$ as above, the set-theoretic sentence $\sigma \equiv (P \cap S) = \emptyset$ is an observational advantage of $\mathcal{D}$ and is semantically entailed by both $\mathcal{D}$ and $\mathcal{S}$.

The motivation for generalizing the concept of a free ride to observational advantages arises from the rich way in which statements can be made. In practice, when representing information, people are allowed to choose any appropriate set of statements. This means there is no guarantee that user-defined semantically equivalent sets of statements ensure that observability is preserved.[6] Through our more general approach, we are able to give insight into the relative observational benefits of one representation, $\hat{\Sigma}$, of information over another, Σ, regardless of whether the statements in one of these sets are all observable from the other.

[6] Such situations can arise in application areas like ontology engineering, where teams of people devise statements in order to define a domain of interest. Team members may have their own preferred notation, such as description logic [1], whereas others may prefer visual approaches, such as VOWL [13] or concept diagrams [24]. But if we can determine what is observable for each representation, and consequently what the potential observational advantages of each representation are, then we can better choose our appropriate representation.

To illustrate this point, consider the set

$$\mathcal{S} = \{P = Q, (P \cap R) = \emptyset\}.$$

In this example, $\mathcal{S}$ will play the role of $\hat{\Sigma}$ in Definition 4. One possible *semantically equivalent* representation of this information is given by $\mathcal{D} = \{d_1, d_2\}$ shown in Figure 5; $\mathcal{D}$ and d, also in Figure 5, are playing the roles of Σ and

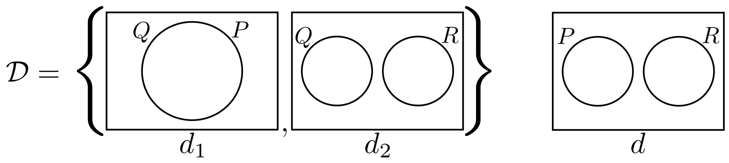


Fig. 5 Observational advantages.

σ respectively, but $\mathcal{D}$ is *not a direct translation* of $\mathcal{S}$. Suppose we wish to know whether P and R represent disjoint sets, expressed by d. The diagram d needs to be inferred from $\mathcal{D}$, since its informational content is not explicitly represented by, and thus not observable from, either d_1 or d_2. By contrast, the informational content of d is explicitly represented in, and thus observable from, $\mathcal{S}$ via $(P \cap R) = \emptyset$. We therefore have:

1. d is not observable from $\mathcal{D}$, and
2. d is observable from $\mathcal{S}$.

Given that we want to know whether P and R represent disjoint sets, $\mathcal{S}$ has an observational advantage over $\mathcal{D}$. Now, similarly to d, the set-theoretic sentence $(P \cap R) = \emptyset$ in $\mathcal{S}$ is not observable from either d_1 or d_2. That is, the set $\mathcal{S}$ *does not* contain only statements observable from $\mathcal{D}$. Thus, according to the definition of free rides, d is not a free ride from $\mathcal{S}$ given $\mathcal{D}$. This implies that the original notion of free rides does not allow us to identify this observational advantage of $\mathcal{S}$ over $\mathcal{D}$. By contrast, d is an observational advantage from $\mathcal{S}$ given $\mathcal{D}$, since $\mathcal{S}$ and $\mathcal{D}$ are semantically equivalent.

Our attention now turns to properties of observational advantages. Suppose we have a finite set, Σ, of statements and a set of statements, $\Sigma_{\models}$, such that Σ semantically entails every statement in $\Sigma_{\models}$; here, $\Sigma_{\models}$ represents the set of conclusions we wish to draw from Σ. It may be that none of the statements in $\Sigma_{\models}$ are observable from Σ. However, an alternative representation of the information in Σ, say by $\hat{\Sigma}$, may ensure that all of the statements in $\Sigma_{\models}$ are observable. If so, every statement in $\Sigma_{\models}$ is an observational advantage of $\hat{\Sigma}$. In this sense, $\Sigma_{\models}$ has *maximal observational advantage.*

Definition 5 Let Σ and $\hat{\Sigma}$ be finite, semantically equivalent sets of statements. Let $\Sigma_{\models}$ be a set of statements such that Σ semantically entails every statement in $\Sigma_{\models}$. If

1. Σ is observationally devoid with respect to $\Sigma_{\models}$, and

2. $\hat{\Sigma}$ is observationally complete with respect to $\Sigma_{\models}$

then $\hat{\Sigma}$ has **maximal observational advantage** over Σ given $\Sigma_{\models}$.

As a trivial example, taking $\Sigma = \mathcal{D}_1 = \{d_1\}$ and $\hat{\Sigma} = \mathcal{D}_2 = \{d_2\}$, where d_1 and d_2 are in Figure 6[7], $\mathcal{D}_2$ has maximal observational advantage over $\mathcal{D}_1$, where $\Sigma_{\models} = \mathcal{D}_{1_{\models}} = \{d_3, d_4\}$.

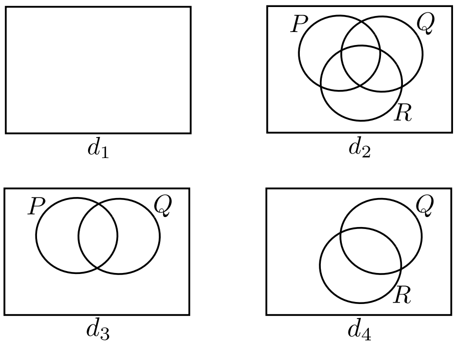


Fig. 6 Maximal observational advantages.

Whenever $\hat{\Sigma}$ has maximal observational advantage, it is a particularly efficacious representation of the information in Σ when we want to deduce all of the statements in $\Sigma_{\models}$. This is because $\hat{\Sigma}$ makes *all* of the statements in $\Sigma_{\models}$ observably true, whereas they must *all* be inferred from Σ (in the sense of using inference rules to make deductions). In the special case where $\Sigma_{\models}$ comprises all of the statements that are semantically entailed by, but not in, Σ, we can think of $\hat{\Sigma}$ as being, from the perspective of observability, an optimal representation of the information in Σ.

5 Two Formal Systems: Set Theory and Euler Diagrams

To develop the theory of observation and observational advantages in the case of set theory and Euler diagrams, we require a formalization of both systems. This section presents typical formalizations. The set theory formalization is commonly known whereas a formalizations for Euler diagrams builds on [25]. Both notations incorporate formal syntax and semantics, adopting a model-theoretic approach. To enable ready comparison of statements made across notations, the set of labels used to denote sets will be common to both set theory and diagrams. Moreover, the interpretation of these labels as sets will also be common. Section 5.1 defines these common entities, with the later subsections defining the statements that can be made, and their semantics, by

[7] Note that d_1 and d_2 are semantically equivalent because they are both true under the same circumstances. Formally, these diagrams have the same *models*, a concept made precise in the next section.

set theory in Section 5.2 and by Euler diagrams in Section 5.3. The last Section 5.4 defines semantic equivalence and entailment across the two notations, crucial for our theory of observational advantages.

5.1 Common Concepts

Both set-theoretic notation and Euler diagrams can be used to make assertions about sets, such as $P \subseteq Q$ and $(P \cup Q) \subseteq R$, visually illustrated by the Euler diagrams d_1 and d_2 respectively in Figure 7. In order to give an account of

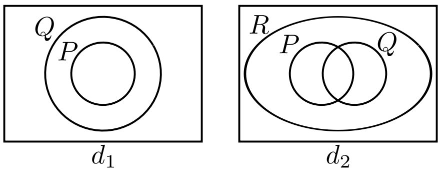


Fig. 7 Visualizing sets using Euler diagrams.

the observational advantages delivered by Euler diagrams over set-theoretic notation, we must provide a common definition of the *interpretation* of the labels, such as P, Q and R, used to represent sets. First, these labels are chosen from a given set.

Definition 6 Define $\mathcal{L}$ to be a set whose elements are called **labels**. Two special symbols, $\emptyset$ and U, are not in $\mathcal{L}$.

Formally, labels *represent* sets and are, thus, interpreted as subsets of some universal set, $\triangle$; we use $\mathbb{P}\triangle$ to denote the powerset of $\triangle$.

Definition 7 An **interpretation** is a pair, $\mathcal{I} = (\triangle, \Psi)$, where $\triangle$ is a set and Ψ is a function, $\Psi\colon \mathcal{L} \cup \{\emptyset, U\} \to \mathbb{P}\triangle$, that maps labels to subsets of $\triangle$ and ensures that $\Psi(\emptyset) = \emptyset$ and $\Psi(U) = \triangle$.

For example, taking $\mathcal{L} = \{P, Q, R\}$, and $\triangle = \{1, 2, 3, 4, 5\}$, we can define $\Psi(P) = \{1, 2\}$, $\Psi(Q) = \{1, 2, 4\}$ and $\Psi(R) = \{1, 5\}$. Then the assertion $P \subseteq Q$ and the diagram d_1 in Figure 7 both express true statements given this interpretation, since $\Psi(P) \subseteq \Psi(Q)$. By contrast, $(P \cup Q) \subseteq R$ and d_2 do not express true statements given this interpretation since $\Psi(P) \cup \Psi(Q) = \{1, 2, 4\} \not\subseteq \{1, 5\} = \Psi(R)$. As we proceed through the next two subsections, we will identify the conditions under which an interpretation 'agrees with' the intended meaning of assertions made about sets using set-theoretic sentences and Euler diagrams.

5.2 Set Theory

Set-theoretic expressions are commonly formed using four operators, $\cap$, $\cup$, $\backslash$, and $^{-}$ (i.e., intersection, union, difference and complement), as follows.

Definition 8 The following are **set-theoretic expressions** or, simply, set-expressions:

1. U and $\emptyset$ are both set-expressions,
2. every label in $\mathcal{L}$ is a set-expression, and
3. if s_1 and s_2 are set-expressions then so are $(s_1 \cap s_2)$, $(s_1 \cup s_2)$, $(s_1 \backslash s_2)$, and $\overline{s_1}$.

We will often blur the distinction between set-expressions and the sets they represent. For instance, strictly speaking $(P \cap Q)$ represents the set $\Psi(P) \cap \Psi(Q)$, given an interpretation $(\triangle, \Psi)$, but when speaking informally we will simply say 'the set $(P \cap Q)$' for simplicity.

Given sets, such as $(P \cup Q)$ and $(P \cap R)$, we are able to assert relationships between sets, such as $(P \cup Q) \subseteq (P \cap R)$, $P = (Q \backslash R)$, $(P \cup Q) = U$, and $(P \cap Q) = \emptyset$. In our case study, we consider the following standard relationships between sets.

Definition 9 Given set-expressions s_1 and s_2 the following are **set-theoretic sentences**:

1. $s_1 \subseteq s_2$, and
2. $s_1 = s_2$.

The set of all set-theoretic sentences is denoted $\mathcal{S}_{\mathcal{ALC}}$.

When we want to give set-expressions names we will use $\equiv$ to avoid overloading $=$ in an ambiguous and potentially confusing way. For example, given the set-expression s_1 and the set-theoretic sentence $s_1 = s_2$, in order to name them s and t respectively, we will write $s \equiv s_1$ and $t \equiv s_1 = s_2$ instead of $s = s_1$ and $t = s_1 = s_2$.

It will be useful, for our work on observation in Section 6, to have access to the labels used in set-expressions.

Definition 10 Let s be a set-expression. The set of **labels in s**, denoted $L(s)$, is defined recursively:

1. if $s \equiv U$ or $s \equiv \emptyset$ then $L(s) = \emptyset$.
2. if $s \in \mathcal{L}$ then $L(s) = \{s\}$,
3. if $s \equiv (s_1 \star s_2)$, where $\star \in \{\cap, \cup, \backslash\}$, then $L(s) = L(s_1) \cup L(s_2)$, and
4. if $s \equiv \overline{s_1}$ then $L(s) = L(s_1)$.

Our attention now turns to formal semantics. We have already defined the notion of an interpretation, which maps labels to sets. Our next step is to extend the mapping, Ψ, so that it interprets more complex set-expressions as sets.

Definition 11 Let s be a set-expression. Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. An extension of Ψ to map set-expressions to sets is defined as follows: for each set-expression, s,

1. if $s \in \mathcal{L} \cup \{U, \emptyset\}$ then $\Psi(s)$ is already defined,
2. if $s \equiv (s_1 \star s_2)$, where $\star \in \{\cap, \cup, \backslash\}$, then $\Psi(s) = \Psi(s_1) \star \Psi(s_2)$, and
3. if $s \equiv \overline{s_1}$ then $\Psi(s) = \overline{\Psi(s_1)} = \Psi(U)\backslash\Psi(s_1)$.

Whenever we have an interpretation, we assume that the extension to set-expressions as just defined is given. We are now in a position to define when an interpretation is a *model* for a set-theoretic sentence.

Definition 12 Let s be a set-theoretic sentence. Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. Then $\mathcal{I}$ **satisfies** s and is a **model** for s under the following circumstances:

1. if $s \equiv s_1 \subseteq s_2$ then $\Psi(s_1) \subseteq \Psi(s_2)$, and
2. if $s \equiv s_1 = s_2$ then $\Psi(s_1) = \Psi(s_2)$.

Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\mathcal{I}$ **satisfies** $\mathcal{S}$ and is a **model** for $\mathcal{S}$ provided $\mathcal{I}$ is a model for each set-theoretic sentence in $\mathcal{S}$.

For example, suppose $\mathcal{L} = \{P, Q, R\}$, and we have $\mathcal{I} = (\triangle, \Psi)$ where $\triangle = \{1, 2, 3, 4, 5\}$, $\Psi(P) = \{1, 2\}$, $\Psi(Q) = \{1, 2, 4\}$ and $\Psi(R) = \{1, 5\}$. Then $P \subseteq Q$ is modelled by $\mathcal{I}$ whereas $(P \cup Q) \subseteq R$ is not. Therefore, taking $\mathcal{S} = \{P \subseteq Q, (P \cup Q) \subseteq R\}$, we can see that $\mathcal{I}$ does not model $\mathcal{S}$.

Lastly, we recap what is meant by a meaning-carrying relationship for set-theoretic sentences. We have already observed that any set-theoretic sentence, s, has only a single meaning-carrying relationship:

- if $s \equiv s_1 \subseteq s_2$ then the meaning-carrying relationship of s is that s_1 is written to the left-hand side of $\subseteq$ and s_2 is written to the right-hand side, and
- if $s \equiv s_1 = s_2$ then the meaning-carrying relationship of s is that s_1 is written to the left-hand side of $=$ and s_2 is written to the right-hand side.

Obviously, these relationships determine the truth-values of set-theoretic sentences in an interpretation, and so they conform to this requirement of meaning carriers given earlier in the paper.

5.3 Euler Diagrams

Euler diagrams are formed from closed curves, each of which has a label drawn from $\mathcal{L}$. We adopt the standard approach of formalizing Euler diagrams via an abstract syntax. This abstract syntax identifies curves with their labels and captures the spatial relationships between the curves via *zones*. In particular, the zones correspond to regions formed from the curves.

To illustrate, the Euler diagram in Figure 8 contains three curves with labels P, Q and R. These curves give rise to six zones. One such zone is outside

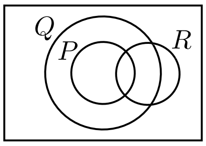


Fig. 8 The syntax of Euler diagrams.

all three curves. Another is inside just the curve labelled Q but outside the remaining two curves. In general, each of the diagram's zones is a region that can be described as being inside some (or none) of the diagram's curves and outside the remaining curves. This insight is used in our formal definition of a zone:

Definition 13 A **zone** is a pair of finite, disjoint sets of labels, (in, out), drawn from $\mathcal{L}$. The set of all zones is denoted $\mathcal{Z}$. A finite set of zones is a **region**. The set of all regions is denoted $\mathcal{R}$.

In terms of drawn Euler diagrams, the labels in the set *in* arise from the curves that the zone is inside whereas the labels in the set *out* arise from the curves that the zone is outside. So, formally, the zone outside all of the curves in Figure 8 is $(\emptyset, \{P, Q, R\})$ and the zone inside just Q is $(\{Q\}, \{P, R\})$. The other four zones are: $(\{P, Q\}, \{R\})$, $(\{P, Q, R\}, \emptyset)$, $(\{Q, R\}, \{P\})$ and $(\{R\}, \{P, Q\})$. Our definition of an Euler diagram records the labels used and the zones that are present.

Definition 14 An **Euler diagram**, d, is a pair, (L, Z), where

1. L is a finite subset of $\mathcal{L}$, and
2. Z is a set of zones such that each (in, out), in Z ensures $in \cup out = L$.

Given $d = (L, Z)$, we sometimes write $L(d)$ and $Z(d)$ for L and Z respectively. The set of all Euler diagrams is denoted $\mathcal{D}_{\mathcal{ALC}}$.

Given an Euler diagram, it is helpful to define the *regions* of the diagram and the *missing zones* of the diagram. Intuitively, the regions are formed from the diagram's zones. Informally, the missing zones (which we define shortly) are zones which could be present given the labels used in the diagram but are not there. Missing zones allow us to define the diagram's semantics.

Definition 15 Let $d = (L, Z)$ be an Euler diagram. The **regions of *d*** are elements of $R(d) = \mathbb{P}Z$.

Definition 16 Let $d = (L, Z)$ be an Euler diagram. The **missing zones of *d*** are elements of $MZ(d) = \{(in, L \backslash in) : in \subseteq L\} \backslash Z$.

For example, consider again d in Figure 5 on page 13. For d, the sets of labels and zones are $L = \{P, R\}$ and $Z = \{(\{P\}, \{R\}), (\{R\}, \{P\}), (\emptyset, \{P, R\})\}$. Now, the set of all possible zones that can be generated from the labels (i.e.,

$\{(in, L\backslash in) : in \subseteq L\}$ is $\{(\{P\},\{R\}),(\{R\},\{P\}),(\emptyset,\{P,R\}),(\{P,R\},\emptyset)\}$. Notice how the intersection of P and R (i.e., the zone $(\{P,R\},\emptyset)$) is not present in the diagram d in Figure 5 – it is a missing zone. Thus, $MZ(d) = \{(\{P,R\},\emptyset)\}$.

We are now in a position to define the semantics of Euler diagrams. Similar to our treatment of set theory, we must first extend interpretations so that they assign sets to zones and regions.

Definition 17 Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. An extension of Ψ to map zones and regions to sets is defined as follows:

1. for each zone, (in, out),
$$\Psi(in, out) = \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)},$$
and
2. for each region, r,
$$\Psi(r) = \bigcup_{(in,out) \in r} \Psi(in, out).$$

Wherever we talk of interpretations, we assume that the extension to zones and regions is given, as well as the extension to set-expressions. Again, just as for our treatment of set theory, we also have the notion of a model for Euler diagrams:

Definition 18 Let $d = (L, Z)$ be an Euler diagram. Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. Then $\mathcal{I}$ **satisfies** d and is a **model** for d whenever $\Psi(MZ(d)) = \emptyset$. Let $\mathcal{D}$ be a finite set of Euler diagrams. Then $\mathcal{I}$ **satisfies** $\mathcal{D}$ and is a **model** for $\mathcal{D}$ provided $\mathcal{I}$ is a model for each Euler diagram in $\mathcal{D}$.

For example, suppose $\mathcal{L} = \{P, Q, R\}$, and we have $\mathcal{I} = (\triangle, \Psi)$ where $\triangle = \{1, 2, 3, 4, 5\}$, $\Psi(P) = \{1, 2\}$, $\Psi(Q) = \{1, 2, 4\}$ and $\Psi(R) = \{1, 5\}$. Then d_1 in Figure 7 is modelled by $\mathcal{I}$ whereas d_2 is not. Therefore, taking $\mathcal{D} = \{d_1, d_2\}$, we can see that $\mathcal{I}$ does not model $\mathcal{D}$.

As with set theory, we now recap the notion of a meaning-carrying relationship for Euler diagrams. We observe that the curves give rise to the meaning of diagrams. When two curves, P and Q, have disjoint interiors, they represent disjoint sets. At the abstract syntax level, this is formally identified by there being no zones inside both curves. That is, the *region*, r, 'inside' both curves is missing from the diagram, identified by the following property:

$$r = \{(in, out) \in Z(d) : P, Q \in in\} = \emptyset.$$

Continuing in this vein, if one curve, R, is contained by another, S, at the abstract syntax level this is captured by the region r_R inside R being contained by the region r_S inside S:

$$r_R = \{(in, out) \in Z(d) : R \in in\} \subseteq \{(in, out) \in Z(d) : S \in in\} = r_S$$

In general, the diagram's *regions* give rise to meaning-carrying relationships. Given an Euler diagram, $d = (L, Z)$, where regions r_1 and r_2 are in d, the following are meaning-carrying relationships of d:

– if $r_1 \subseteq r_2$ then $r_1 \subseteq r_2$ is a meaning-carrying relationship, and
– if $r_1 = r_2$ then $r_1 = r_2$ is a meaning-carrying relationship.

Notice that other relationships between regions, such as $r_1 \cap r_2 \neq \emptyset$ in d are not meaning carriers. In the given instance, whilst $\Psi(r_1) \cap \Psi(r_2) \neq \emptyset$ evaluates to either true or false in any interpretation, in some models for d the assertion $\Psi(r_1) \cap \Psi(r_2) \neq \emptyset$ will be true and in others it will be false. A clear requirement on meaning carriers is that they must have a fixed interpretation in models for d. That is, they cannot be true in some models and false in others.

5.4 Semantic Equivalence and Entailment

Our last key task in this section is to formally define semantic relationships between statements, regardless of whether they are set-theoretic sentences or Euler diagrams. To illustrate, taking $\mathcal{S} = \{P \subseteq Q, (P \cup Q) \subseteq R\}$ and $\mathcal{D} = \{d_1, d_2\}$, where d_1 and d_2 are in Figure 7 on page 15, the set-theoretic sentences and the Euler diagrams have the same models. This means that they are *semantically equivalent*. To formally define semantic equivalence, and the related notion of *semantic entailment*, we first define the generic notion of a statement.

Definition 19 A **statement** is defined as follows:

1. any set-theoretic sentence is a statement, and
2. any Euler diagram is a statement.

We will typically denote Euler diagrams by d_i, set-theoretic sentences by s_i and statements in general by σ_i. Similarly, we will denote a set of Euler diagrams by $\mathcal{D}$, a set of set-theoretic sentences by $\mathcal{S}$ and a set of statements by Σ. Now, the notion of a model for a set of statements naturally generalizes the definitions of a model already given: an interpretation is a model for Σ if it is a model for every statement in Σ. At this point, we are able to define semantic equivalence.

Definition 20 Let σ_1 and σ_2 be statements. If σ_1 and σ_2 have the same models then they are **semantically equivalent**. Let Σ_1 and Σ_2 be finite sets of statements. If Σ_1 and Σ_2 have the same models then they are **semantically equivalent**.

Definition 21 Let Σ be a finite set of statements and let σ be a statement. Then Σ **semantically entails** σ, denoted $\Sigma \vDash \sigma$, provided every model for Σ is also a model for σ. If σ is semantically entailed by, but not in Σ, then σ is **non-trivially** semantically entailed by Σ.

For example, given $\mathcal{S} = \{(P \cap Q) = \emptyset, R \subseteq P, R \subseteq T\}$ and $s_1 \equiv (R \cap Q) = \emptyset$, we have $\mathcal{S}$ semantically entails $(P \cap Q) = \emptyset$, $R \subseteq P, R \subseteq T$ and s_1. In the case of s_1, which is also non-trivially semantically entailed, this is because whenever the set-theoretic sentences in $\mathcal{S}$ are true, it must also be the case that s_1 is true. By contrast, $\mathcal{S}$ does not semantically entail $s_2 \equiv P = Q$. That is, we can pick the interpretation $\mathcal{I} = (\{1,2\}, \Psi)$ where $\Psi(P) = \{1\}$, $\Psi(Q) = \{2\}$, $\Psi(R) = \{1\}$, and $\Psi(T) = \{1,2\}$ is a model for $\mathcal{S}$ but not for s_2 (notably, $\Psi(P) \neq \Psi(Q)$).

Considering Figure 9, given $\mathcal{D} = \{d', d''\}$ and d_1, $\mathcal{D}$ semantically entails d', d'' and d_1. In the case of d_1, which is also non-trivially semantically entailed, this is because whenever the Euler diagrams in $\mathcal{D}$ are true, it must also be the case that d_1 is true. By contrast, $\mathcal{D}$ does not semantically entail d_2. That is, we can pick the interpretation $\mathcal{I} = (\{1,2\}, \Psi)$ where $\Psi(P) = \{1\}$, $\Psi(Q) = \{2\}$, $\Psi(R) = \{1\}$, and $\Psi(T) = \{1,2\}$ is a model for $\mathcal{D}$ but not for d_2 (notably, the missing zone $(\{P\}, \{Q\})$ does not represent the empty set: $\Psi(\{P\}, \{Q\}) = \{1\}$).

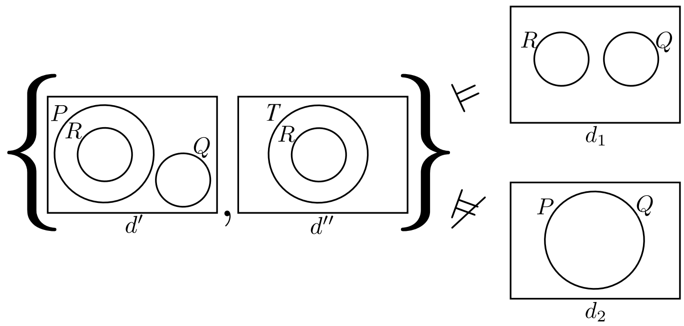


Fig. 9 Semantic entailment.

The next definition provides us with notation and terminology with which to talk about the set of statements that are semantically entailed by $\mathcal{S}$.

Definition 22 Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Define $\mathcal{S}_{\models}$ to be the set of set-theoretic sentences that are non-trivially semantically entailed by $\mathcal{S}$. Let L be a set of labels. We define $\mathcal{S}_{\models}^{L}$ to be the largest subset of $\mathcal{S}_{\models}$ such that every set-theoretic sentence, s, in $\mathcal{S}_{\models}^{L}$ ensures $L(s) \subseteq L$.

For example, taking $\mathcal{S} = \{(P \cup Q) \subseteq R, (P \cap Q) = \emptyset\}$, the set-theoretic sentences $P \subseteq R$ and $Q \subseteq R$ are both in $\mathcal{S}_{\models}^{\{P,Q,R\}}$ (along with many others), whereas $((P \cap T) \cap Q) = \emptyset$ uses the label T and, so, is not in $\mathcal{S}_{\models}^{\{P,Q,R\}}$ despite being in $\mathcal{S}_{\models}$.

One can readily define similar notions for Euler diagrams.

Definition 23 Let $\mathcal{D}$ be a finite set of Euler diagrams. Define $\mathcal{D}_{\models}$ to be the set of Euler diagrams that are non-trivially semantically entailed by $\mathcal{D}$. Let L be a set of labels. We define $\mathcal{D}_{\models}^{L}$ to be the largest subset of $\mathcal{D}_{\models}$ such that every Euler diagram, d, in $\mathcal{D}_{\models}^{L}$ ensures $L(d) \subseteq L$.

In Figure 10, taking $\mathcal{D} = \{d\}$, the set $\mathcal{D}_{\models}$ contains all diagrams with no missing zones (i.e., Venn diagrams), other than d. The set $\mathcal{D}_{\models}^{L(d)}$ contains all Euler diagrams in $\mathcal{D}_{\models}$ whose labels are chosen from $\{P, Q, R\}$.

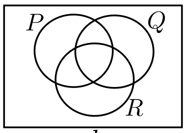


Fig. 10 Semantic entailment of sets of diagrams.

6 Observation Relations

We now have a method of comparing semantics across these two notations, so we are in a position to define what it means for a statement made in one of them to be *observable* from a statement made in the other. For our purposes, we also need to identify when one set-theoretic sentence is observable from another. We need these three observation relations because they allow us to establish that for any set of set-theoretic sentences, say $\mathcal{S}$, there exists a diagram, d, such that for any set-theoretic sentence, s, that is non-trivially semantically entailed by $\mathcal{S}$:

1. s is not observable from $\mathcal{S}$, for which we need an observation relation on set-theoretic sentences, and
2. s is observable from $\{d\}$, for which we need an observation relation between diagrams and set-theoretic sentences.

In addition, to identify such a diagram, d, we first define an observation relation between set-theoretic sentences and diagrams. This permits us to translate each set-theoretic sentence, s', in $\mathcal{S}$ into an observable diagram, d'. After this process, all of the diagrams resulting from this translation can be unified to produce the required d.

6.1 Observing set-theoretic sentences from set-theoretic sentences

Any meaning-carrying relationship in a set-theoretic sentence is determined by the whole statement, thus each set-theoretic sentence has a unique meaning-carrying relationship. Moreover, the only set-theoretic sentence observable from a set-theoretic sentence is the set-theoretic sentence itself.

Definition 24 Let s_1 and s_2 be set-theoretic sentences. Then s_2 is **observable** from s_1, denoted $s_1 \rightsquigarrow s_2$, iff s_1 and s_2 are the same set-theoretic sentence.

Trivially, we have the following lemma which is stated because it is a requirement that the observation relation respects semantic entailment:

Lemma 2 *Let s_1 and s_2 be set-theoretic sentences. If $s_1 \rightsquigarrow s_2$ then s_1 semantically entails s_2.*

6.2 Observing Euler diagrams from set-theoretic sentences

We now define what it means for a diagram d to be observable from a set-theoretic sentence, s. Since each set-theoretic sentence has a unique meaning-carrying relationship, there is a *unique* observable diagram d from s. To exemplify this point, consider the set-theoretic sentence $s \equiv (P \cup Q) \subseteq R$. The unique meaning-carrying relationship in s can be described in the following way: the pair of labels P and Q connected by the symbol $\cup$ are connected to label R with the symbol $\subseteq$ and no other visuo-spatial relationship holds among these and other labels in $\mathcal{L}$. Call this relationship δ. The meaning carried by δ is that P union Q is a subset of R. Now, the visuo-spatial relationship carrying this meaning in Euler diagrams is that the union of the curves labelled P and Q is included in the curve labelled R. Call this relationship γ. Since δ is the only meaning carrier in s, the only diagram observable from s is the diagram that supports γ and supports only those spatial relationships that are necessary to express the meaning carried by δ and nothing stronger. For example, consider Figure 11: the diagram d_1 is just that and is observable from s[8]. By contrast, neither d_2 nor d_3 are observable from s. Lacking a curve labelled Q, the diagram d_2 fails to support γ. The diagram d_3, whilst semantically equivalent to s, is not observable from s, since it supports unnecessary relationships involving the curve labelled S.

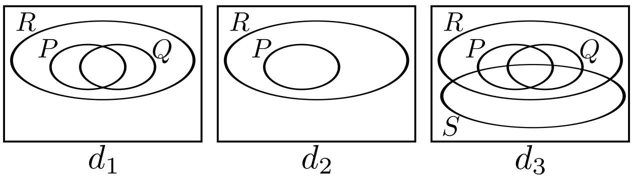


Fig. 11 Observing diagrams from set-theoretic sentences.

To formally define the diagram observable from a set-theoretic sentence, s, we need access to the labels used in s; these are the labels that must be present in the observable diagram. Once we have access to them, we can convert set-expressions to regions from which a diagram will be built. We will exemplify the approach using $s \equiv (P \cup Q) \subseteq R$ as a running example, formed from the set-expressions $s_1 \equiv (P \cup Q)$ and $s_2 \equiv R$. Given $L(s_1) = \{P, Q\}$ and $L(s_2) = \{R\}$, we need to create regions that comprise zones that are formed over $\{P, Q, R\}$; these are the labels to be present in the required diagram and no others. The core idea is to translate each set-expression, s_i, given a set of

[8] One may wonder if the partial overlap of the curves labelled P and Q in d_1 is really 'necessary', since even if these curves stood in a different relationship (e.g., separated), the diagram could support γ and express the information that P union Q is a subset of R. However, any of such alternative relationships would make the diagram express something stronger than this information (e.g., that P and Q are disjoint). Having the partial overlap of the curves labelled P and Q is necessary in this sense—it is the only way to translate s to an Euler diagram without expressing anything stronger.

labels, L, to a region, r_i, that represents the same set as s_i. Moreover, the zones in r_i must each form a two-way partition of L. The set L must include all labels in $L(s)$ (of course, as L contains labels it must also be a subset of $\mathcal{L}$). In our example, the region r_1 for s_1 is *required* to represent a subset of r_2, the region for s_2. This is because the diagram must convey the meaning-carrying relationship $s_1 \subseteq s_2$.

Definition 25 Let s be a set-expression and let L be a set of labels such that $L(s) \subseteq L$. The **translation** of s given L into a region, denoted $\mathcal{T}(s, L)$, is defined recursively:

1. if $s \equiv \emptyset$ then $\mathcal{T}(s, L) = \emptyset$,
2. if $s \equiv U$ then $\mathcal{T}(s, L) = \{(in, out) \in \mathcal{Z} : in \cup out = L\}$,
3. if $s \in \mathcal{L}$ then $\mathcal{T}(s, L) = \{(in, out) \in \mathcal{T}(U, L) : s \in in\}$,
4. if $s \equiv (s_1 \star s_2)$, where $\star \in \{\cap, \cup, \backslash\}$, then $\mathcal{T}(s, L) = (\mathcal{T}(s_1, L) \star \mathcal{T}(s_2, L))$, and
5. if $s \equiv \overline{s_1}$ then $\mathcal{T}(s, L) = (\mathcal{T}(U, L) \backslash \mathcal{T}(s_1, L))$.

In our running example, taking $L = L(s_1) \cup L(s_2) = \{P, Q, R\}$, we get:

1. the translation of $s_1 \equiv P \cup Q$ is given by

$$\begin{aligned}
\mathcal{T}(s_1, L) &= \mathcal{T}(P \cup Q, L)\\
&= \mathcal{T}(P, L) \cup \mathcal{T}(Q, L)\\
&= \{(in, out) \in \mathcal{T}(U, L) : P \in in\} \cup \{(in, out) \in \mathcal{T}(U, L) : Q \in in\}\\
&= \{(\{P\}, \{Q, R\}), (\{P, Q\}, \{R\}), (\{P, R\}, \{Q\}), (\{P, Q, R\}, \emptyset)\} \cup\\
&\quad \{(\{Q\}, \{P, R\}), (\{P, Q\}, \{R\}), (\{Q, R\}, \{P\}), (\{P, Q, R\}, \emptyset)\}\\
&= \{(\{P\}, \{Q, R\}), (\{P, Q\}, \{R\}), (\{P, R\}, \{Q\}), (\{P, Q, R\}, \emptyset),\\
&\quad (\{Q\}, \{P, R\}), (\{Q, R\}, \{P\})\},
\end{aligned}$$

2. the translation of $s_2 \equiv R$ is given by

$$\begin{aligned}
\mathcal{T}(s_2, L) &= \mathcal{T}(R, L)\\
&= \{(in, out) \in \mathcal{T}(U, L) : R \in in\}\\
&= \{(\{R\}, \{P, Q\}), (\{P, R\}, \{Q\}), (\{Q, R\}, \{P\}), (\{P, Q, R\}, \emptyset)\}.
\end{aligned}$$

An important property of our translation is that it preserves semantics. To illustrate, consider R and $\mathcal{T}(R, L)$ just given and an arbitrary interpretation $\mathcal{I} = (\triangle, \Psi)$:

$$\begin{aligned}
\Psi(R) &= (\Psi(R) \cap \overline{\Psi(P)}) \cup (\Psi(R) \cap \Psi(P))\\
&= (\Psi(R) \cap \overline{\Psi(P)} \cap \overline{\Psi(Q)}) \cup (\Psi(R) \cap \overline{\Psi(P)} \cap \Psi(Q)) \cup\\
&\quad (\Psi(R) \cap \Psi(P) \cap \overline{\Psi(Q)}) \cup (\Psi(R) \cap \Psi(P) \cap \Psi(Q))\\
&= \Psi(\{R\}, \{P, Q\}) \cup \Psi(\{Q, R\}, \{P\}) \cup \Psi(\{P, R\}, \{Q\}) \cup\\
&\quad \Psi(\{P, Q, R\}, \emptyset)\\
&= \Psi(\mathcal{T}(R, L)).
\end{aligned}$$

Lemma 3 *Let s be a set-expression and let L be a set of labels such that $L(s) \subseteq L$. Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. Then $\Psi(s) = \Psi(\mathcal{T}(s, L))$.*

To construct a diagram, d_1, for $s_1 \subseteq s_2$, we need to ensure that in any model, $(\triangle, \Psi)$, for d_1, $\Psi(\mathcal{T}(s_1, L(s_1) \cup L(s_2))) \subseteq \Psi(\mathcal{T}(s_2, L(s_1) \cup L(s_2)))$. To achieve this, the missing zones of d_1 are those in $\mathcal{T}(s_1, L(s_1) \cup L(s_2))$ that are not in $\mathcal{T}(s_2, L(s_1) \cup L(s_2))$. By making these zones missing from d_1, we ensure that they each represent the empty set in any model for d_1, thus ensuring that $\Psi(\mathcal{T}(s_1, L(s_1) \cup L(s_2))) \subseteq \Psi(\mathcal{T}(s_2, L(s_1) \cup L(s_2)))$. This means that d_1 will be semantically equivalent to $s_1 \subseteq s_2$. Thus, our diagram d_1 has a missing zone set:

$$MZ(d_1) = \{(\{P\}, \{Q, R\}), (\{P, Q\}, \{R\}), \{Q\}, \{P, R\})\}.$$

This is the missing zone set for d_1 in Figure 11 on page 23, which we have already seen is observable from $(P \cup Q) \subseteq R$. For set-theoretic sentences of the form $s_1 = s_2$, the missing zones are those in the symmetric difference of $\mathcal{T}(s_1, L(s_1) \cup L(s_2))$ and $\mathcal{T}(s_2, L(s_1) \cup L(s_2))$, because these zones must represent empty sets in order for

$$\Psi(\mathcal{T}(s_1, L(s_1) \cup L(s_2))) = \Psi(\mathcal{T}(s_2, L(s_1) \cup L(s_2)))$$

to hold and, therefore, $\Psi(s_1) = \Psi(s_2)$, to be true.

Finally, we are able to define when a diagram d is observable from a set relation s.

Definition 26 Let s be a set-theoretic sentence and let d be an Euler diagram. Then d is **observable** from s, denoted $s \rightsquigarrow d$, iff

1. if s is of the form $s_1 \subseteq s_2$ then
   (a) $L(d) = L(s_1) \cup L(s_2)$, and
   (b) $MZ(d) = \mathcal{T}(s_1, L(d)) \backslash \mathcal{T}(s_2, L(d))$ and
2. if s is of the form $s_1 = s_2$ then
   (a) $L(d) = L(s_1) \cup L(s_2)$, and
   (b) $MZ(d) = \big(\mathcal{T}(s_1, L(d)) \backslash \mathcal{T}(s_2, L(d))\big) \cup \big(\mathcal{T}(s_2, L(d)) \backslash \mathcal{T}(s_1, L(d))\big)$.

Again, we state the following lemma since the observation relation must respect semantic entailment.

Lemma 4 *Let s be a set-theoretic sentence and let d be an Euler diagram. If $s \rightsquigarrow d$ then s semantically entails d.*

In fact, if $s \rightsquigarrow d$ then s and d are semantically equivalent.

6.3 Observing set-theoretic sentences from Euler diagrams

Our attention now turns to defining when a set-theoretic sentence, s, is observable from an Euler diagram, d. For s to be observable, the unique meaning-carrying relationship of s must arise from a meaning-carrying relationship in d. Now, the curves in Euler diagrams give rise to regions. The relationships between these regions are meaning-carrying and they visually indicate the relationships between the sets they represent. For example, in Figure 12, $r_1 \subseteq r_2$ where

$$r_1 = \{(\{P, Q\}, \{R\}), (\{P, Q, R\}, \emptyset)\}$$

and

$$r_2 = \{(\{P, Q\}, \{R\}), (\{P, Q, R\}, \emptyset), (\{Q\}, \{P, R\}), (\{Q, R\}, \{P\})\}.$$

Semantically, in any model, $\mathcal{I} = (\triangle, \Psi)$, for this diagram we have $\Psi(r_1) \subseteq \Psi(r_2)$. To identify the set-theoretic sentences observable from a diagram, we need to consider how we describe the regions in diagrams and to what those descriptions correspond in terms of set-expressions.

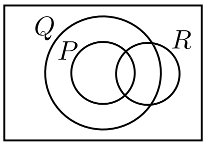


Fig. 12 Representing regions.

Given a region in an Euler diagram, there are many different ways of describing it. For example, in Figure 12, the region r_1 just given above can be described in at least the following ways:

1. r_1 is the region inside P,
2. r_1 is the region inside both P and Q,
3. r_1 is the region inside P but outside R combined with the region inside both P and R, and
4. r_1 is the region that does not include anything outside of P.

These ways of describing regions can be formally captured by describing sets of zones in the region via the curves those zones lie inside and the curves they lie outside. For instance, saying that r_1 is the region inside P can be more precisely stated as 'r_1 is the region completely inside P and not completely outside any of the curves'. A set of zones with a description such as this is called a *zonal region*, since the description is akin to how we describe zones [9]. Regions can then be described through different ways of combining zonal regions. For instance, in our example, r_1 can also be described using two zonal regions: *the region completely inside* P *and completely outside* R union *the region completely inside both* P *and* R *and not completely outside any of the curves.*

It is possible to completely describe any region using combinations of zonal regions and there are many different ways of doing so. To make this notion of describing regions precise, we first define zonal regions.

Definition 27 Let $d = (L, Z)$ be an Euler diagram. Let IN and OUT be finite disjoint sets of labels such that $IN \cup OUT \subseteq L$. The **zonal region over *L* in *d***, denoted $\langle IN, OUT \rangle$, is defined to be

$$\langle IN, OUT \rangle = \{(in, out) \in Z : IN \subseteq in \wedge OUT \subseteq out\}.$$

For example, given $d = (L, Z)$, where $L = \{P, Q, R, S\}$ and

$$Z = \{(in, out) \in \mathcal{Z} : in \cup out = L\},$$

taking $IN = \{P, Q\}$ and $OUT = \{R\}$, we have

$$\langle \{P, Q\}, \{C\} \rangle = \{(\{P, Q, S\}, \{R\}), (\{P, Q\}, \{R, S\}\}.$$

We now define region descriptions, which are merely syntactic devices used to characterize different ways in which regions can be built from zonal regions in a diagram, d. Note that descriptions of regions are not unique.

Definition 28 Let $d = (L, Z)$ be an Euler diagram and let r be a region of d. A **description** of r in d is defined as follows:

1. if $r = \emptyset$ then $\emptyset$ is a description of r in d.
2. if r is a zonal region of the form $\langle IN, OUT \rangle$ then the tuple $\langle IN, OUT \rangle$ is a description of r in d,
3. if $r = r_1 \star r_2$ for some regions r_1 and r_2 in d and $\star \in \{\cap, \cup, \backslash\}$ then a description of r is $(R_1 \star R_2)$ where R_1 and R_2 are descriptions of r_1 and r_2 in d, respectively, and
4. if $r = \overline{r_1}$ for some region r_1 then a description of r is $\overline{R_1}$ where R_1 is a description of r_1 in d.

Returning to Figure 12, the region, r_1, comprising the zones $(\{P, Q\}, \{R\})$ and $(\{P, Q, R\}, \emptyset)$ can be formally described in at least the following ways:

1. $\langle \{P\}, \emptyset \rangle$,
2. $\langle \{P, Q\}, \emptyset \rangle$
3. $\langle \{P\}, \{R\} \rangle \cup \langle \{P, R\}, \emptyset \rangle$, and
4. $\overline{\langle \emptyset, \{P\} \rangle}$.

Trivially, we have the following result:

Lemma 5 *Let d be an Euler diagram. Every region of d has a description in d.*

Our next task is to turn region descriptions into set-expressions. Consider a zone, (in, out). Then, by Definition 17, (in, out) is interpreted as the set

$$\Psi(in, out) = \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)},$$

in any interpretation, $\mathcal{I} = (\triangle, \Psi)$. Thus, (in, out) readily corresponds to a set-expression of the form (noting the omission of brackets, which are formally required):

$$\bigcap_{l \in in} l \cap \bigcap_{l \in out} \bar{l}.$$

We can use this insight to define set-expressions arising from zonal regions. For instance, the zonal region $\langle\{P\},\{R\}\rangle$ represents $(P \cap \overline{R})$ and, equivalently, $(\overline{R} \cap P)$. Since the diagram prescribes no reading order for the sets, we can take both $(P \cap \overline{R})$ and $(\overline{R} \cap P)$ as being set-expressions arising from $\langle\{P\},\{R\}\rangle$ reflecting the commutativity of $\cap$. The set-expressions arising from zonal regions also need to reflect the associativity of $\cap$. For instance, the zonal region $\langle\{P,Q\},\{R,S\}\rangle$ can be translated into the following set-expressions, amongst others:

1. $(P \cap (Q \cap (\overline{R} \cap \overline{S})))$,
2. $((P \cap Q) \cap (\overline{R} \cap \overline{S}))$, and
3. $((P \cap (Q \cap \overline{R})) \cap \overline{S})$.

The rich way of translating zonal regions into set-expressions is captured by Definition 29, part 2. Before presenting this definition, we illustrate the idea more precisely by showing the steps required to translate $\langle\{P,Q\},\{R,S\}\rangle$ into $((P \cap Q) \cap (\overline{R} \cap \overline{S}))$. In particular, we have

$$\begin{aligned}\langle\{P,Q\},\{R,S\}\rangle &= \langle\{P,Q\},\emptyset\rangle \cap \langle\emptyset,\{R,S\}\rangle \\ &= (((\langle\{P\},\emptyset\rangle \cap \langle\{Q\},\emptyset\rangle) \cap (\langle\emptyset,\{R\}\rangle \cap \langle\emptyset,\{S\}\rangle)) \\ &\mapsto ((P \cap Q) \cap (\overline{R} \cap \overline{S})).\end{aligned}$$

In the last (translation) step $\mapsto$, zonal regions of the form $\langle\{l\},\emptyset\rangle$ and $\langle\emptyset,\{l\}\rangle$ are simply replaced by l and $\bar{l}$ respectively.

Definition 29 Let d be an Euler diagram and let r be a region of d. Let s be a set-expression. We say that s is a **translation** of r in d iff there exists a description, R, of r in d where one of the following four conditions holds:

1. $R = \emptyset$ and $s \equiv \emptyset$.
2. $R = \langle IN, OUT\rangle$ and either
   (a) $IN \cup OUT = \emptyset$ and $s \equiv U$,
   (b) $IN \cup OUT = \{l\}$ and $s \equiv l$, where $l \in IN$,
   (c) $IN \cup OUT = \{l\}$ and $s \equiv \bar{l}$, where $l \in OUT$, or
   (d) $|IN \cup OUT| \geq 2$ and there exist two disjoint, non-empty subsets, L_1 and L_2, of $IN \cup OUT$ such that
      i. $L_1 \cup L_2 = IN \cup OUT$,
      ii. there exists a set-expression, s_1, that is a translation of
      $$\langle IN \backslash L_1, OUT \backslash L_1\rangle,$$
      iii. there exists a set-expression, s_2, that is a translation of
      $$\langle IN \backslash L_2, OUT \backslash L_2\rangle,$$
      and
      iv. $s \equiv (s_1 \cap s_2)$.

3. $R = (R_1 \star R_2)$, for some region descriptions R_1 and R_2, and there exist set-expressions s_1 and s_2 where s_1 and s_2 that are translations of R_1 and R_2 respectively such that $s \equiv (s_1 \star s_2)$.
4. $R = \overline{R_1}$ for some region description R_1 and there exists a translation s_1 of R_1 such that $s \equiv \overline{s_1}$.

Importantly, in any model for d, the set represented by a region is the same as that represented by any translation of it.

Lemma 6 *Let d be an Euler diagram and let r be a region of d. Let s be a set-expression that is a translation of r in d. Let $\mathcal{I} = (U, \Psi)$ be a model for d. Then $\Psi(s) = \Psi(r)$.*

We are now in a position to identify the set-theoretic sentences observable from diagrams. Recall that the relationship between two regions, r_1 and r_2, in an Euler diagram d is meaning-carrying and corresponds to a relationship between the represented sets. If $r_1 \subseteq r_2$ then we can observe $s_1 \subseteq s_2$ where s_1 and s_2 are translations of r_1 and r_2 respectively. Similarly, if $r_1 = r_2$ then $s_1 = s_2$ is observable. For example, in Figure 12 we have already seen that $r_1 \subseteq r_2$ where

$$r_1 = \{(\{P, Q\}, \{R\}), (\{P, Q, R\}, \emptyset)\}$$

and

$$r_2 = \{(\{P, Q\}, \{R\}), (\{P, Q, R\}, \emptyset), (\{Q\}, \{P, R\}), (\{Q, R\}, \{P\})\}.$$

The regions r_1 and r_2 can be described by $\langle\{P\}, \emptyset\rangle$ and $\langle\{Q\}, \emptyset\rangle$ respectively. Therefore, from Figure 12 we can observe $P \subseteq Q$.

Definition 30 Let d be an Euler diagram and let $s_1 \star s_2$, where $\star \in \{\subseteq, =\}$, be a set-theoretic sentence. Then $s_1 \star s_2$ is **observable** from d, denoted $d \rightsquigarrow s_1 \star s_2$, provided there exist regions r_1 and r_2 of d such that

1. $r_1 \star r_2$,
2. s_1 is a translation of r_1, and
3. s_2 is a translation of r_2.

To conclude this section, we establish that this observation relation respects semantic entailment.

Lemma 7 *Let d be an Euler diagram and let s be a set-theoretic sentence. If $d \rightsquigarrow s$ then d semantically entails s.*

7 Observational Completeness and Devoidness

The purpose of this section is to demonstrate that Euler diagrams are particularly powerful representations of information about sets, as compared to set-theoretic sentences, from the perspectives of observational completeness and devoidness.

7.1 Observational Devoidness of set-theoretic sentences

Firstly we establish that set-theoretic sentences are observationally devoid representations of information. Referring to Definition 1 on page 10, given a finite set of set-theoretic sentences, $\mathcal{S}$, and a set-theoretic sentence s, we have: s is observable from $\mathcal{S}$, denoted $\mathcal{S} \rightsquigarrow s$, iff there is an s_i in $\mathcal{S}$ such that $s_i \rightsquigarrow s$. By Definition 3 on page 10, as the only set-theoretic sentences observable from $\mathcal{S}$ are in $\mathcal{S}$, we have the following theorem.

Theorem 1 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\mathcal{S}$ is observationally devoid with respect to $\mathcal{S}_{\models}$.*

Theorem 1 tells us that if we want to deduce any set-theoretic sentence from $\mathcal{S}$ that is not explicitly written down in $\mathcal{S}$ then we must use inference mechanisms to deduce it. In this sense, set-theoretic sentences are weak representations of information.

7.2 Observational Completeness of Euler Diagrams

Our attention now turns to identifying conditions under which Euler diagrams are observationally complete representations of information, relative to set-theoretic sentences. Suppose we have a finite set of set-theoretic sentences, $\mathcal{S} = \{s_1, ..., s_n\}$ and we take $\mathcal{D} = \{d_i \in \mathcal{D}_{\mathcal{ALL}} : \exists s_i \in \mathcal{S}\ s_i \rightsquigarrow d_i\}$. Then, given a single s_i and its associated d_i, any set-theoretic sentence that follows from s_i, up to labels, is observable from d_i. In other words, by Definition 2 on page 10, $\{d_i\}$ is observationally complete with respect to $\{s_i\}_{\models}^{L(s_i)}$ (i.e., the set of set-theoretic sentences that follow semantically from $\{s_i\}$ whose labels are all in $L(s_i)$).

As a specific example, consider $\mathcal{S} = \{s_1 \equiv P = Q, s_2 \equiv (Q \cap R) = \emptyset\}$ and the associated set $\mathcal{D}$ illustrated in Figure 5 on page 13. Whilst we can observe $P = Q$ and $(Q \cap R) = \emptyset$ from $\mathcal{D}$, we *cannot* observe the set-theoretic sentence $(P \cap R) = \emptyset$ from either d_1 or d_2, just like we cannot observe the diagram d from d_1 or d_2. Thus, in this case $\mathcal{D}$ is *not* observationally complete with respect to $\mathcal{S}$. In general, if we take $\mathcal{S}$ to be a finite set of set-theoretic sentences with cardinality at least two, it is *not* necessarily the case that $\mathcal{D} = \{d \in \mathcal{D}_{\mathcal{ALL}} : \exists s \in \mathcal{S}\ s \rightsquigarrow d\}$ is observationally complete with respect to $\mathcal{S}_{\models}^{L}$, where $L = \bigcup_{s \in \mathcal{S}} L(s)$. However, we will establish that for every finite set of set-theoretic sentences, $\mathcal{S}$, there is a single Euler diagram, d, that *is* observationally complete with respect to $\mathcal{S}_{\models}^{L}$.

To illustrate, taking $\mathcal{S} = \{s_1 \equiv P \subseteq Q, s_2 \equiv (Q \cap R) = \emptyset, s_3 \equiv T \subseteq R\}$ an observationally complete Euler diagram is d in Figure 13; $\mathcal{D} = \{d\}$ is semantically equivalent to $\mathcal{S}$. Here, any set-theoretic sentence, s, that is semantically entailed by $\mathcal{S}$ and that is formed using only labels that are used in set-theoretic sentences in $\mathcal{S}$ is observable from $\mathcal{D}$; for example, taking $s \equiv (P \cap R) = \emptyset$ we can observe s from $\mathcal{D}$ (although we cannot observe s from $\mathcal{S}$).

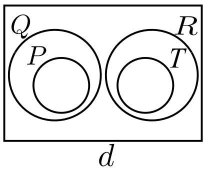


Fig. 13 Observational completeness.

The fact that all such statements, s, are observable from $\mathcal{D}$ means that $\mathcal{D}$ is observationally complete with respect to $\mathcal{S}$, and also with respect to all of the set-theoretic sentences that are non-trivially semantically entailed by $\mathcal{S}$, up to the labels used.

In order to build an observationally complete diagram, d, from $\mathcal{S}$, we start by identifying, for each s_i in $\mathcal{S}$, the Euler diagram d_i observable from it. In our example, these diagrams are shown in Figure 14. Then we unify these diagrams into a single diagram, firstly unifying d_1 and d_2 and secondly unifying the result of this first operation with d_3. The resulting diagram is d, as originally seen in Figure 13.

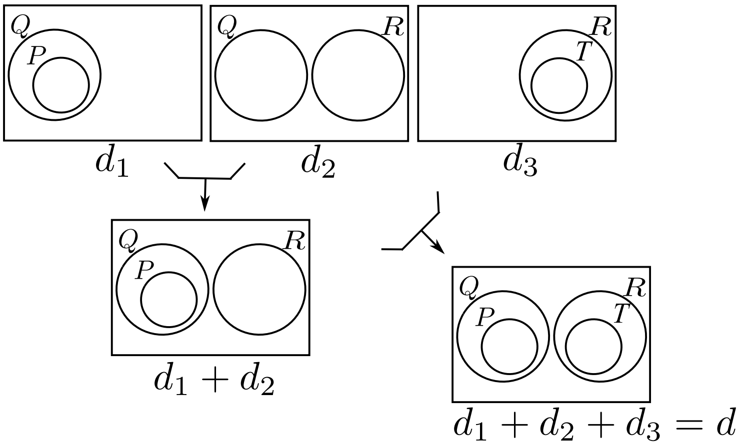


Fig. 14 Building diagrams from set-theoretic sentences.

Definition 31 Let $d_1 = (L_1, Z_1)$ and $d_2 = (L_2, Z_2)$ be Euler diagrams. The **unification** of d_1 and d_2, denoted $d_1 + d_2$, is an Euler diagram, $d_1 + d_2 = (L_1 + L_2, Z_1 + Z_2)$ where

1. the labels are given by $L_1 + L_2 = L_1 \cup L_2$, and
2. the zones are given by

$$\begin{aligned} Z_1 + Z_2 = \{&(in \cup L, out \cup (L_2 \backslash (L_1 \cup L))) : (in, out) \in Z_1 \wedge L \subseteq L_2 \backslash L_1\} \cap \\ \{&(in \cup L, out \cup (L_1 \backslash (L_2 \cup L))) : (in, out) \in Z_2 \wedge L \subseteq L_1 \backslash L_2\}. \end{aligned}$$

An essential property of the unification operation is that it preserves semantics. That is, if we unify d_1 and d_2 then $\{d_1, d_2\}$ is semantically equivalent to $\{d_1 + d_2\}$, captured by Lemma 8.

Lemma 8 *Let $d_1 = (L_1, Z_1)$ and $d_2 = (L_2, Z_2)$ be Euler diagrams. Then $\{d_1, d_2\}$ is semantically equivalent to $\{d_1 + d_2\}$.*

It is also useful to note that unification is a commutative and associative operation. This means that the result of converting each set-theoretic sentence in $\mathcal{S}$ into a diagram and then unifying them is not dependent on the order in which this is carried out: there is a unique resulting diagram.

Lemma 9 *Let $d_1 = (L_1, Z_1)$, $d_2 = (L_2, Z_2)$ and $d_3 = (L_3, Z_3)$ be Euler diagrams. Then*

1. *$d_1 + d_2 = d_2 + d_1$ (the operation + is commutative), and*
2. *$(d_1 + d_2) + d_3 = d_1 + (d_2 + d_3)$ (the operation + is associative).*

Definition 32 Let $\mathcal{S} = \{s_1, ..., s_n\}$ be a finite set of set-theoretic sentences. The Euler diagram $d = d_1 + ... + d_n$, where for each i we have $s_i \rightsquigarrow d_i$, is the **unified** diagram for $\mathcal{S}$. The unified diagram for $\mathcal{S}$ is denoted $d_{\mathcal{S}}$.

The diagram $d_{\mathcal{S}}$ is semantically equivalent to $\mathcal{S}$.

Lemma 10 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\mathcal{S}$ and $\{d_{\mathcal{S}}\}$ are semantically equivalent.*

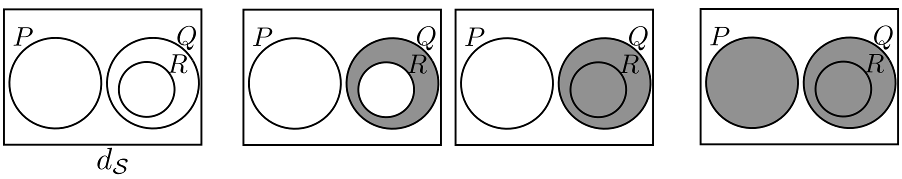


Fig. 15 Observing set-theoretic sentences.

To complete this section, we show that the unified diagram, $d_{\mathcal{S}}$, for $\mathcal{S}$ is observationally complete. The proof of this result starts by showing that for any set-expression formed over the labels in $L = \bigcup_{s \in \mathcal{S}} L(s)$ there is a region in $d_{\mathcal{S}}$ that translates to s. Then given a set-theoretic sentence, say $s' \equiv s_1 \subseteq s_2$, in $\mathcal{S}^L_{\models}$, we know that for both set-expressions s_1 and s_2 there are regions, r_1 and r_2 respectively, in $d_{\mathcal{S}}$ that translate to s_1 and s_2 (the same is true if, instead, $s' \equiv s_1 = s_2$). Our task is to show that $r_1 \subseteq r_2$, for then we can observe that $s_1 \subseteq s_2$.

To illustrate, consider $\mathcal{S} = \{P \cap Q = \emptyset, R \subseteq Q\}$. The unified diagram for $\mathcal{S}$ is shown as $d_{\mathcal{S}}$ in Figure 15. The set-theoretic sentence $(Q \backslash R) \subseteq Q$ is in $\mathcal{S}^L_{\models}$ and is formed from two set-expressions, namely $(Q \backslash R)$ and Q. The regions for these two set-expressions are also shown in Figure 15 in the middle two diagrams (we use shading to point them out). Comparing these shaded regions, it is easy to see that $r_1 \subseteq r_2$. Therefore, since r_1 translates to $Q \backslash R$ and r_2 translates to Q, we have shown that $(Q \backslash R) \subseteq Q$ is observable from $d_{\mathcal{S}}$. Similarly, $P \cup Q$ is a translation of the shaded region in the rightmost diagram. Therefore, we can also observe that $R \subseteq (P \cup Q)$ from $d_{\mathcal{S}}$.

Theorem 2 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\{d_{\mathcal{S}}\}$ is observationally complete with respect to $\mathcal{S}^L_{\vDash}$ where $L = \bigcup\limits_{s \in \mathcal{S}} L(s)$.*

Theorem 2 tells us that a single Euler diagram is a powerful representation of information as compared to using set-theoretic sentences. This is because *every* set-theoretic sentence that must be inferred from $\mathcal{S}$, up to labels, can simply be observed from the Euler diagram.

8 Observational Advantages of Euler Diagrams

As discussed in Section 2, diagrams often have a number of observational advantages over sentential notations when representing information. Using our theory of observation and the results in Section 7, we are able to characterize these advantages. In particular, we have seen the following:

1. Theorem 1: a set of set-theoretic sentences, $\mathcal{S}$, is observationally devoid with respect to $\mathcal{S}^L_{\vDash}$, where $L = \bigcup\limits_{s \in \mathcal{S}} L(s)$, and
2. Theorem 2: the (set containing the) unified Euler diagram d is observationally complete with respect to $\mathcal{S}^L_{\vDash}$.

In particular, *any* statement σ, be it a set-theoretic sentence or Euler diagram, that is semantically entailed by $\mathcal{S}$ and not in $\mathcal{S}$ *must be inferred* from $\mathcal{S}$. By contrast, σ *can be observed* from d. In the specific case of set-theoretic sentences compared to diagrams, we have the following theorem.

Theorem 3 *Let $\mathcal{S} = \{s_1, ..., s_n\}$ be a finite set of set-theoretic sentences. Then $\mathcal{D} = \{d_{\mathcal{S}}\}$ has maximal observational advantage over $\mathcal{S}$ given $\mathcal{S}^L_{\vDash}$ where $L = \bigcup\limits_{s \in \mathcal{S}} L(s)$.*

Thus, a single Euler diagram is the most efficacious representation of the information in $\mathcal{S}$. Consequently, we showed that a single diagram to represent information has observational advantages compared to using sentential systems. By contrast, a set of diagrams that does not include the unified diagram need not have maximal observational advantage over $\mathcal{S}$.

9 Discussion and Conclusion

Theoretically characterizing relative advantages of representations of information allows us to choose our representation in an informed way. Through the notion of observation introduced in this paper, we are able to provide such a characterization. In particular, it is *advantageous* if a representation of information allows us to simply observe other statements of interest to be true. By contrast, if we cannot observe the statement – yet it does indeed follow from the given representation – then this is a *disadvantage* of that representation. If one representation of information, r_1, has such an advantage and another, r_2, has this as a disadvantage then r_1 has an *observational advantage* over r_2.

Through demonstrating how to apply our theory of observation and observational advantages to set theory and Euler diagrams, we have been able to establish the following:

1. Given a finite set of set-theoretic sentences, $\mathcal{S}$, no other set-theoretic sentences can be observed from $\mathcal{S}$; thus, $\mathcal{S}$ is *observationally devoid.*
2. Given an Euler diagram, $d_{\mathcal{S}}$, constructed from $\mathcal{S}$, every set-theoretic sentence that follows from $\mathcal{S}$ can be observed from $d_{\mathcal{S}}$; thus, $d_{\mathcal{S}}$ is *observationally complete.*

These two characterizations of what can (or cannot) be observed allow us to understand that $d_{\mathcal{S}}$ is a significantly more efficacious representation of information than $\mathcal{S}$: it has *maximal observational advantage* over $\mathcal{S}$. From a theoretical perspective, these benefits mean that using $d_{\mathcal{S}}$ is desirable: $d_{\mathcal{S}}$ makes informational content readily available, in the sense of observability, to end-users; the discussion below on *net cognitive value* considers this point from the perspective of human cognition. As there are infinitely many set-theoretic sentences that are semantically entailed by $\mathcal{S}$, the benefits of Euler diagrams over set theory are numerous. Linking back to the insight that a diagram is sometimes worth 10,000 words, our formal theory of observation and observational advantage has allowed us to prove that a diagram is sometimes worth *infinitely many* set-theoretic sentences.

In our view, this result captures the kernel in which diagrammatic representations facilitate our inference and thus excel over sentential representations. By putting this result in a larger perspective, we may expect to gain a fuller understanding of the relative advantages of one choice of representation over another where diagrams are one of the options. We pose two general questions that lead to extensions of our research.

The first question is concerned with the generalizability of our result. When, generally, do we find a unifying diagram that is observationally complete with respect to a set of sentential representations? It clearly depends on the choice of our diagrammatic and sentential notations, and so on the kinds of information we want to express with our notations. In this paper, we have investigated the case where we are only interested in expressing the subset relation and the equality relation among sets. What happens when we are also interested in the expression of non-subsetness and non-equality? For example, we may wish to express $P \not\subseteq Q$ and $P \neq R$. Diagrammatically, this can be expressed by augmenting Euler diagrams with $\otimes$-sequences, originally introduced by Peirce [15] and further explored by Shin [20]; $\otimes$-sequences assert the non-emptiness of sets. Figure 16 demonstrates how to express $P \not\subseteq Q$ and $P \neq R$. It will be interesting to determine whether and when these augmented Euler diagrams still have maximal observational advantages.

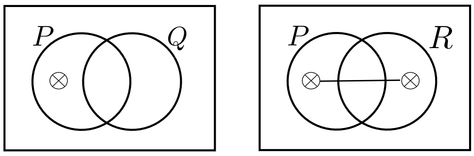


Fig. 16 Augmenting Euler diagrams with $\otimes$ to assert non-emptiness of sets.

Diagrammatic notations need not always have maximal observational advantages over sentential notations, and it is important to investigate the syntactic and semantic conditions of a pair of notations under which this happens — that is, we should identify when the diagrammatic notation is guaranteed to have a unifying diagram over any set of statements in the sentential notation. This way we will see more clearly how, exactly, diagrammatic notations work in the kernel cases.

Still widening our perspective, the second question is concerned with the *net cognitive value* of observability. *Observing* statements from sets of statements comes with cognitive cost, just as *inferring* statements from sets of statements does. For example, d_1 in Figure 17 is the unified diagram of the set of statements:

$$\mathcal{S} = \{(P \cap V) = \emptyset, (P \cap T) = \emptyset, (Q \cap V) = \emptyset, (Q \cap T) = \emptyset, (R \cap T) = \emptyset, T \subseteq V\},$$

and hence every set-theoretic sentence semantically entailed by $\mathcal{S}$ is observable from d_1 as far as it is concerned with the labels $\{P, Q, R, T, V\}$ only. In particular, $((P \backslash R) \cap ((V \backslash R) \backslash T)) = \emptyset$ is observable, and since it is not a member of $\mathcal{S}$, it is an observational advantage of the diagram d_1 over $\mathcal{S}$. Observing

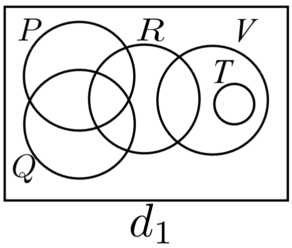


Fig. 17 Cost of observing a statement from a diagram.

this statement, however, requires one to recognize the two regions in d_1 corresponding to the set-expressions $(P \backslash R)$ and $((V \backslash R) \backslash T)$ as well as their spatial relationship, while inferring it from $\mathcal{S}$ seems to take only a few short inference steps: trivially, $(P \backslash R) \subseteq P$ and $((V \backslash R) \backslash T) \subseteq V$, so $((P \backslash R) \cap ((V \backslash R) \backslash T)) = \emptyset$ follows from $(P \cap V) = \emptyset$.

Thus, the net cognitive value of a statement observable from a diagram depends both on the cost of recognizing relevant diagrammatic elements and their spatial relationships, and also on the set of available observational operations with which one may reach that statement from a set of set-theoretic sentences[9]. Therefore, our study of observational advantages, conducted purely on a logical basis, must be connected in the future to *psychological* and *computational* models of the *perceptual* as well as *observational* operations available to users.

With such a connection, we may expect to compare, more generally, two semantically equivalent sets of representations, where each has observational advantages over the other. To compute these advantages, we will need to evaluate the observational as well as the perceptual costs of each representation (with respect to the set $\Sigma_{\models}$ of information relevant to the cognitive task at hand). Our logical approach presented in this paper that formalises the distinction between what is observable and what must be inferred from a given set of representations is the crucial first step in such a connected approach to the general representation comparison problem.

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[9] Another type of cost would arise if the diagram from which observation is made was not just given, but had to be constructed. Imagine how you would construct a diagram given the premises $(P \cup Q) \subseteq R$ and $R \subseteq P$. Once a diagram is successfully constructed, it lets you observe interesting consequences such as $P = Q$ and $Q = R$, but the construction of the diagram must have required a significant effort on your part in figuring out how to enclose regions on a plane with curves to satisfy a set of containment conditions. Such a 'construction cost' would have to be weighed if we were to consider not just a given Euler diagram and a given set of set-theoretic sentences, but also their construction processes out of a set of premises. We thank one of our reviewers for the suggestion of this example.

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What Makes an Effective Representation of Information: A Formal Account of Observational Advantages Appendix

Gem Stapleton · **Mateja Jamnik** · **Atsushi Shimojima**

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Abstract This document presents proofs of the lemmas (Appendix A) and theorems (Appendix B) contained within the associated paper.

A Proofs of the Lemmas

Lemma 1 *Let Σ and $\hat{\Sigma}$ be finite, semantically equivalent sets of statements. Let σ be an observational advantage of $\hat{\Sigma}$ given Σ. Then Σ semantically entails σ.*

Proof Since σ is an observational advantage, it can be observed from $\hat{\Sigma}$. This means that there is a statement, $\hat{\sigma}$, in $\hat{\Sigma}$ such that σ can be observed from $\hat{\sigma}$. Therefore, since observation relations must respect semantic entailment, $\hat{\sigma}$ semantically entails σ. This implies that $\hat{\Sigma}$ also semantically entails σ. Since $\hat{\Sigma}$ and Σ are semantically equivalent, Σ must also semantically entail σ. □

Lemma 2 *Let s_1 and s_2 be set-theoretic sentences. If $s_1 \rightsquigarrow s_2$ then s_1 semantically entails s_2.*

Proof This result trivially holds since s_1 and s_2 are the same set-theoretic sentence. □

G. Stapleton
University of Brighton
Brighton, UK
E-mail: g.e.stapleton@brighton.ac.uk

M. Jamnik
University of Cambridge
Cambridge, UK
E-mail: mateja.jamnik@cl.cam.ac.uk

A. Shimojima
Doshisha University
Kyoto, Japan
E-mail: ashimoji@mail.doshisha.ac.jp

Lemma 3 *Let s be a set-expression and let L be a set of labels such that $L(s) \subseteq L$. Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. Then $\Psi(s) = \Psi(\mathcal{T}(s, L))$.*

Proof The proof is by induction over the depth of s in the inductive construction of set-expressions. There are three base cases:

- Case 1: $s \equiv \emptyset$. In this case, $\mathcal{T}(s, L) = \emptyset$. Trivially $\Psi(s) = \Psi(\mathcal{T}(s, L))$ and we are done.
- Case 2: $s \equiv U$. In this case, $\mathcal{T}(s, L) = \{(in, out) \in \mathcal{Z} : in \cup out = L\}$. We know that $\Psi(U) = \triangle$, by definition. Therefore, we must show $\Psi(\mathcal{T}(s, L)) = \triangle$. Firstly, it is obvious that $\Psi(\mathcal{T}(s, L)) \subseteq \triangle$. Let $e \in \triangle$. We show that $e \in \Psi(in, out)$ for some $(in, out) \in \mathcal{T}(s, L)$. To begin, we note that for each $l \in L$, either $e \in \Psi(l)$ or $e \notin \Psi(l)$. Define $in = \{l \in L : e \in \Psi(l)\}$ and $out = \{l \in L : e \notin \Psi(l)\}$. Clearly, $(in, out) \in \mathcal{T}(s, L)$. Moreover,
$$e \in \bigcap_{l \in in} \Psi(l) \qquad \text{and} \qquad e \in \bigcap_{l \in out} \overline{\Psi(l)}.$$
So
$$e \in \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)}.$$
Therefore, $e \in \Psi(in, out)$, so $e \in \Psi(\mathcal{T}(s, L))$. Hence $\Psi(\mathcal{T}(s, L)) = \triangle$, as required.
- Case 3: $s \in \mathcal{L}$. In this case, $\mathcal{T}(s, L) = \{(in, out) \in \mathcal{T}(U, L) : s \in in\}$. We must show $\Psi(s) = \Psi(\mathcal{T}(s, L))$. Let $e \in \triangle$ and, to show that $\Psi(s) \subseteq \Psi(\mathcal{T}(s, L))$, suppose that $e \in \Psi(s)$. We identify a zone in $\mathcal{T}(s, L)$ whose interpretation under Ψ includes e. As in Case 2, for each $l \in L$, either $e \in \Psi(l)$ or $e \notin \Psi(l)$. Define $in = \{l \in L : e \in \Psi(l)\}$ and $out = \{l \in L : e \notin \Psi(l)\}$. Clearly, in includes s (since $e \in \Psi(s)$) and, therefore, $(in, out) \in \mathcal{T}(s, L)$. Moreover,
$$e \in \bigcap_{l \in in} \Psi(l) \qquad \text{and} \qquad e \in \bigcap_{l \in out} \overline{\Psi(l)}.$$
From this it follows that
$$e \in \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)} = \Psi(in, out).$$
Therefore, $e \in \Psi(in, out)$, so $e \in \Psi(\mathcal{T}(s, L))$. Thus, $\Psi(s) \subseteq \Psi(\mathcal{T}(s, L))$.
To show that $\Psi(\mathcal{T}(s, L)) \subseteq \Psi(s)$, suppose instead that $e \in \Psi(\mathcal{T}(s, L))$. Choose the zone (in, out) in $\mathcal{T}(s, L)$ such that $e \in \Psi(in, out)$. By definition,
$$\Psi(in, out) = \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)}.$$
Since $(in, out) \in \mathcal{T}(s, L)$ it follows that $s \in in$, so we deduce that
$$\Psi(in, out) = \bigcap_{l \in in} \Psi(l) \cap \bigcap_{l \in out} \overline{\Psi(l)} \cap \Psi(s).$$
Therefore $\Psi(in, out) \subseteq \Psi(s)$, so $e \in \Psi(s)$. Hence $\Psi(\mathcal{T}(s, L)) \subseteq \Psi(s)$ and it follows that $\Psi(\mathcal{T}(s, L)) = \Psi(s)$, as required.

Hence the three base cases hold.

Suppose for any set-expression, s, at depth n in the inductive construction of set-expressions, that $\Psi(s) = \Psi(\mathcal{T}(s, L))$. Let s' be a set-expression at depth $n+1$. Then either $s' \equiv (s_1 \star s_2)$ or $s' \equiv \overline{s_1}$. In the former case, by assumption we have $\Psi(s_1) = \Psi(\mathcal{T}(s_1, L))$ and $\Psi(s_2) = \Psi(\mathcal{T}(s_2, L))$. Therefore,

$$\begin{aligned}\Psi(s') &= \Psi(s_1) \star \Psi(s_1)\\ &= \Psi(\mathcal{T}(s_1, L)) \star \Psi(\mathcal{T}(s_2, L))\\ &= \Psi(\mathcal{T}(s_1, L) \star \mathcal{T}(s_2, L))\\ &= \Psi(\mathcal{T}(s', L)).\end{aligned}$$

The case where $s' \equiv \overline{s_1}$ is similar. Hence, $\Psi(s) = \Psi(\mathcal{T}(s, L))$, for all set-expression s. □

Lemma 4 *Let s be a set-theoretic sentence and let d be an Euler diagram. If $s \rightsquigarrow d$ then s semantically entails d.*

Proof Let $\mathcal{I} = (\triangle, \Psi)$ be a model for s. We must show that $\mathcal{I}$ is a model for $d = (L, Z)$. There are two cases.

- Case 1: $s \equiv s_1 \subseteq s_2$. By Lemma 3, we know that $\Psi(s_1) = \Psi(\mathcal{T}(s_1, L))$ and $\Psi(s_2) = \Psi(\mathcal{T}(s_2, L))$. Since $\mathcal{I}$ is a model for s, we further know that

$$\Psi(\mathcal{T}(s_1, L)) \subseteq \Psi(\mathcal{T}(s_2, L)). \quad (1)$$

  Now, to show that $\mathcal{I}$ is a model for d, we must show that every missing zone of d represents the empty set under Ψ. Let z be a missing zone of d. Then, by Definition 27,

$$z \in \mathcal{T}(s_1, L) \backslash \mathcal{T}(s_2, L). \quad (2)$$

  If $\Psi(z) \neq \emptyset$ then (1) implies that there is a zone, z', in $\mathcal{T}(s_2, L(d))$ such that $\Psi(z) \cap \Psi(z') \neq \emptyset$. It is easy to show that distinct zones in $Z(d) \cup MZ(d)$ represent disjoint sets, so it follows that $z = z'$. But then $z \in \mathcal{T}(s_2, L(d))$, contradicting (2). Thus, $\Psi(z) = \emptyset$. Hence $\mathcal{I}$ is a model for d. Therefore s semantically entails d.
- Case 2: $s \equiv s_1 = s_2$. The proof is similar to Case 1.

Hence, in either case, s semantically entails d. □

Lemma 5 *Let d be an Euler diagram. Every region of d has a description in d.*

Proof If $r = \emptyset$ then $\emptyset$ is a description of r. Otherwise, $r \neq \emptyset$, so r is of the form $r = \{(in_1, out_1), ..., (in_n, out_n)\}$. Each zone (in_i, out_i) is described by $\langle in_i, out_i \rangle$. Then r has description

$$(...((\langle in_1, out_1 \rangle \cup \langle in_2, out_2 \rangle) \cup \langle in_3, out_3 \rangle)... \cup \langle in_n, out_n \rangle)$$

as required. □

Lemma 6 *Let d be an Euler diagram and let r be a region of d. Let s be a set-expression that is a translation of r in d. Let $\mathcal{I} = (U, \Psi)$ be a model for d. Then $\Psi(s) = \Psi(r)$.*

Proof Our proof proceeds by induction on the depth of region descriptions in the inductive construction of such descriptions. There are two base cases.

- Case 1: $R = \emptyset$. In this case, $r = \emptyset$ and $s \equiv \emptyset$, so $\Psi(r) = \emptyset = \Psi(s)$.
- Case 2: $R = \langle IN, OUT \rangle$. There are four subcases.
  (a) $IN \cup OUT = \emptyset$. In this case, $s \equiv U$. Moreover,

$$\langle IN, OUT \rangle = Z(d).$$

It is easy to show that, in any model for d, $\Psi(Z(d)) = \triangle$. Since $\Psi(s) = \Psi(U) = \triangle$ we are done.

  (b) $IN \cup OUT = \{l\}$, where $l \in IN$. In this case, $s \equiv l$. Moreover,

$$\langle IN, OUT \rangle = \{(in, out) \in Z(d) : l \in in\}.$$

Clearly $\Psi(\langle IN, OUT \rangle) \subseteq \Psi(l)$. Let $e \in \Psi(l)$. Then there is a zone, $(in, L\backslash in)$, in $Z(d) \cup MZ(d)$ such that $e \in \Psi(in, L\backslash in)$. From this, it follows that $l \notin L\backslash in$, so $l \in in$. Moreover, since $e \in \Psi(in, L\backslash in)$, we see that $\Psi(in, L\backslash in) \neq \emptyset$. Thus, the zone $(in, L\backslash in)$ is not missing from d. Therefore, $(in, L\backslash in)$ is in $\langle IN, OUT \rangle$. Hence $e \in \Psi(\langle IN, OUT \rangle)$. In conclusion, $\Psi(\langle IN, OUT \rangle) = \Psi(l) = \Psi(s)$.

  (c) $IN \cup OUT = \{l\}$, where $l \in OUT$. This case is similar to (b).
  (d) $|IN \cup OUT| \geq 2$. In this case, there exist two disjoint, non-empty subsets L_1 and L_2 of $IN \cup OUT$ such that
    (a) $L_1 \cup L_2 = IN \cup OUT$,
    (b) there exists a set-expression, s_1, that is a translation of

$$\langle IN\backslash L_1, OUT\backslash L_1 \rangle,$$

    (c) there exists a set-expression, s_2, that is a translation of

$$\langle IN\backslash L_2, OUT\backslash L_2 \rangle,$$

    and
    (d) $s \equiv (s_1 \cap s_2)$.

Noting that $IN = (IN\backslash L_1) \cup (IN\backslash L_2)$ and $OUT = (OUT\backslash L_1) \cup (OUT\backslash L_2)$, it follows that

$$\langle IN, OUT \rangle = \langle IN\backslash L_1, OUT\backslash L_1 \rangle \cap \langle IN\backslash L_2, OUT\backslash L_2 \rangle.$$

Therefore

$$\Psi(\langle IN, OUT \rangle) = \Psi(\langle IN\backslash L_1, OUT\backslash L_1 \rangle) \cap \Psi(\langle IN\backslash L_2, OUT\backslash L_2 \rangle).$$

It can be shown, using induction, that $\Psi(s_1) = \langle IN\backslash L_1, OUT\backslash L_1 \rangle$, $\langle IN\backslash L_2, OUT\backslash L_2 \rangle$. Therefore, $\Psi(\langle IN, OUT \rangle) = \Psi(s_1) \cap \Psi(s_2) = \Psi(s)$, as required.

Assume for any region description, R, at depth n, and for any region, r, with that description, it is the case that $\Psi(r) = \Psi(s)$. For our inductive step we again have two cases. Let r' be a region with a description, R', at depth $n+1$.

- Case 1: $R' = R_1 \star R_2$. Then there exist regions r_1 and r_2 with descriptions R_1 and R_2 respectively such that $r' = r_1 \star r_2$. Then s', the translation of $r_1 \star r_2$, is $s' \equiv (s_1 \star s_2)$ where s_1 and s_2 are translations of R_1 and R_2 respectively. By assumption, $\Psi(r_1) = \Psi(s_1)$ and $\Psi(r_2) = \Psi(s_2)$. Therefore, since $\Psi(r) = \Psi(r_1) \star \Psi(r_2)$ we deduce that $\Psi(r) = \Psi(s_1) \star \Psi(s_2)$.
- Case 2: $R' = \overline{R_1}$. Then there exists a region r_1 with description R_1 such that $r = \overline{r_1}$. Then $s' \equiv \overline{s_1}$ where s_1 is a translation of R_1. By assumption, $\Psi(r_1) = \Psi(s_1)$. Therefore, $\Psi(r) = \overline{\Psi(r_1)} = \overline{\Psi(s_1)}$.

Hence, $\Psi(r) = \Psi(s)$ in all models for d. □

Lemma 7 *Let d be an Euler diagram and let s be a set-theoretic sentence. If $d \rightsquigarrow s$ then d semantically entails s.*

Proof Let $\mathcal{I} = (\triangle, \Psi)$ be a model for d. Let s be a set-theoretic sentence observable from d. Then $s \equiv s_1 \subseteq s_2$ or $s \equiv s_1 = s_2$. In either case, there exist regions r_1 and r_2 such that s_1 and s_2 are the translations of r_1 and r_2 respectively. When $s \equiv s_1 \subseteq s_2$, we know that $r_1 \subseteq r_2$. Therefore, $\Psi(r_1) \subseteq \Psi(r_2)$. By Lemma 6, $\Psi(s_1) \subseteq \Psi(s_2)$. The proof is similar when $s \equiv s_1 = s_2$. □

Lemma 8 *Let $d_1 = (L_1, Z_1)$ and $d_2 = (L_2, Z_2)$ be Euler diagrams. Then $\{d_1, d_2\}$ is semantically equivalent to $\{d_1 + d_2\}$.*

Proof Let $\mathcal{I} = (\triangle, \Psi)$ be an interpretation. Suppose $\mathcal{I}$ is a model for $\{d_1, d_2\}$. Then $\mathcal{I}$ is a model for both d_1 and d_2. We need to show that $\mathcal{I}$ is a model for $d_1 + d_2$. Let $(in, out) \in MZ(d_1 + d_2)$. Suppose, for a proof by contradiction, that $\Psi(in, out) \neq \emptyset$. Let $e \in \Psi(in, out)$. Since $\mathcal{I}$ models d_1, there is a zone, z, in $Z(d_1)$ such that $e \in \Psi(z)$. Choose such a zone, say (in_1, out_1). Since $in_1 \cup out_1 = L_1 \subseteq L_1 \cup L_2 = in \cup out$, and both (in, out) and (in_1, out_1) represent a set containing e, we have

$$in_1 \subseteq in \qquad \text{and} \qquad out_1 \subseteq out.$$

This implies that there exists an L where $L \subseteq (L_2 \backslash L_1)$ such that

$$(in, out) = (in_1 \cup L, out_1 \cup (L_2 \backslash (L_1 \cup L)))$$

Similarly, there is a zone, (in_2, out_2) in $Z(d_2)$ and an L' such that

$$(in, out) = (in_2 \cup L', out_2 \cup (L_1 \backslash (L_2 \cup L')))$$

From this, it follows that $(in, out) \in Z_1 + Z_2 = Z(d_1 + d_2)$, which is a contradiction. Hence our assumption that $\Psi(in, out) \neq \emptyset$ was false. Thus, zones missing from $d_1 + d_2$ represent empty sets. Hence $\mathcal{I}$ is a model for $d_1 + d_2$.

For the converse, suppose that $\mathcal{I}$ is a model for $d_1 + d_2$. We must show that $\mathcal{I}$ models d_1 and d_2. First, we show that $\mathcal{I}$ models d_1. Let (in, out) be a zone in $MZ(d_1)$. We must show $\Psi(in, out) = \emptyset$. Suppose that $\Psi(in, out) \neq \emptyset$ and choose $e \in \Psi(in, out)$. This implies there is a zone, say (in_{1+2}, out_{1+2}), in $Z_1 + Z_2$ (i.e. in $Z(d_1 + d_2)$) such that $e \in \Psi(in_{1+2}, out_{1+2})$. Now, because (in_{1+2}, out_{1+2}) is in Z_1+Z_2, there exists a zone, say (in_1, out_1), in $Z_1 = Z(d_1)$ and a set of labels, L, such that

$$(in_{1+2}, out_{1+2}) = (in_1 \cup L, out_1 \cup (L_2 \backslash (L_1 \cup L))).$$

We therefore know that

$$\begin{aligned} e &\in \Psi(in_1 \cup L, out_1 \cup (L_2 \backslash (L_1 \cup L))) \\ &= \bigcap_{l \in in_1 \cup L} \Psi(l) \cap \bigcap_{l \in out_1 \cup (L_2 \backslash (L_1 \cup L))} \overline{\Psi(l)}. \end{aligned}$$

Therefore,

$$e \in \bigcap_{l \in in_1 \cup L} \Psi(l) \subseteq \bigcap_{l \in in_1} \Psi(l)$$

and

$$e \in \bigcap_{l \in out_1 \cup (L_2 \backslash (L_1 \cup L))} \overline{\Psi(l)} \subseteq \bigcap_{l \in out_1} \overline{\Psi(l)}.$$

This implies that

$$e \in \bigcap_{l \in in_1} \Psi(l) \cap \bigcap_{l \in out_1} = \Psi(in_1, out_1).$$

Since distinct zones in $Z(d_1) \cup MZ(d_1)$ represent disjoint sets, it follows that

$$(in_1, out_1) = (in, out).$$

But (in_1, out_1) is in $Z(d_1)$, meaning that (in, out) is in $Z(d_1)$ and is not in $MZ(d_1)$, which is a contradiction. Therefore, it cannot be that $\Psi(in, out) \neq \emptyset$, so it follows that $\Psi(in, out) = \emptyset$. Since (in, out) was an arbitrary missing zone, we deduce that $\mathcal{I}$ is a model for d_1. The case for d_2 is similar. Hence, $\mathcal{I}$ models both d_1 and d_2. Therefore, $\{d_1, d_2\}$ and $\{d_1 + d_2\}$ are semantically equivalent. □

Before we prove Lemma 9, we establish some properties of zones in unified diagrams, presented here in a new lemma.

Lemma 11 *Let $d_1 = (L_1, Z_1)$ and $d_2 = (L_2, Z_2)$ be Euler diagrams.*

1. *If $(in_z, out_z) \in Z_1 + Z_2$ then $(in_z \cap L_1, out_z \cap L_1) \in Z_1$ and $(in_z \cap L_2, out_z \cap L_2) \in Z_2$.*
2. *If (in_z, out_z) is a zone such that*
   (a) $in_z \cup out_z = L_1 \cup L_2$,
   (b) $(in_z \cap L_1, out_z \cap L_1) \in Z_1$ and
   (c) $(in_z \cap L_2, out_z \cap L_2) \in Z_2$
   then $(in_z, out_z) \in Z_1 + Z_2$.

Proof For the first part of the proof we begin by showing that $(in_z \cap L_1, out_z \cap L_1) \in Z_1$. Since $(in_z, out_z) \in Z_1 + Z_2$ we know that

$$(in_z, out_z) \in \{(in \cup L, out \cup (L_2 \backslash (L_1 \cup L))) : (in, out) \in Z_1 \wedge L \subseteq L_2 \backslash L_1\}.$$

Choose $(in, out) \in Z_1$ and L such that

$$(in_z, out_z) = (in \cup L, out \cup (L_2 \backslash (L_1 \cup L))).$$

Then $in_z = in \cup L$, so

$$in_z \cap L_1 = (in \cup L) \cap L_1.$$

We know that $L \subseteq L_2 \backslash L_1$, so $L \cap L_1 = \emptyset$. Therefore,

$$in_z \cap L_1 = in \cap L_1.$$

Moreover, we also know that $in \subseteq L_1$, because $(in, out) \in Z_1$, thus:

$$in_z \cap L_1 = in.$$

It can also be shown that $out_z \cap L_1 = out$. Therefore,

$$(in_z \cap L_1, out_z \cap L_1) = (in, out) \in Z_1$$

as required. Similarly, $(in_z \cap L_2, out_z \cap L_2) \in Z_2$, completing the first part of the proof.

For the second part, we assume that there is a zone (in_z, out_z) such that

(a) $in_z \cup out_z = L_1 \cup L_2$,
(b) $(in_z \cap L_1, out_z \cap L_1) \in Z_1$ and
(c) $(in_z \cap L_2, out_z \cap L_2) \in Z_2$

We show that $(in_z, out_z) \in Z_1 + Z_2$. Given $(in_z \cap L_1, out_z \cap L_1)$, we take $L = (L_2 \backslash L_1) \cap in_z$. Then

$$\begin{aligned}
(in_z \cap L_1) \cup L &= (in_z \cap L_1) \cup ((L_2 \backslash L_1) \cap in_z) \\
&= ((in_z \cap L_1) \cup in_z) \cap ((in_z \cap L_1) \cup (L_2 \backslash L_1)) \\
&= in_z \cap ((in_z \cup (L_2 \backslash L_1)) \cap (L_1 \cup (L_2 \backslash L_1))) \\
&= in_z \cap ((in_z \cup (L_2 \backslash L_1)) \cap (L_1 \cup L_2)) \\
&= in_z \cap (in_z \cup (L_2 \backslash L_1)) \\
&= in_z.
\end{aligned}$$

That is, $(in_z \cap L_1) \cup L = in_z$. It can also be shown that $out_z \cup (L_2 \backslash (L_1 \cup L)) = out_z$. Therefore, since $L = (L_2 \backslash L_1) \cap in_z \subseteq L_2 \backslash L_1$, we have

$$(in_z, out_z) \in \{(in \cup L', out \cup (L_2 \backslash (L_1 \cup L'))) : (in, out) \in Z_1 \wedge L' \subseteq L_2 \backslash L_1\}.$$

Likewise,

$$(in_z, out_z) \in \{(in \cup L', out \cup (L_1 \backslash (L_2 \cup L'))) : (in, out) \in Z_2 \wedge L' \subseteq L_1 \backslash L_2\}.$$

Therefore $(in_z, out_z) \in Z_1 + Z_2$, as required. □

Lemma 9 *Let $d_1 = (L_1, Z_1)$, $d_2 = (L_2, Z_2)$ and $d_3 = (L_3, Z_3)$ be Euler diagrams. Then*

1. *$d_1 + d_2 = d_2 + d_1$ (the operation + is commutative), and*
2. *$(d_1 + d_2) + d_3 = d_1 + (d_2 + d_3)$ (the operation + is associative).*

Proof The commutativity of + is obvious. Regarding associativity, firstly it is trivial that $(d_1 + d_2) + d_3$ and $d_1 + (d_2 + d_3)$ have the same label set, namely $L(d_1) \cup L(d_2) \cup L(d_3)$. Let (in, out) be a zone of $(d_1 + d_2) + d_3$, that is $(in, out) \in (Z_1 + Z_2) + Z_3$. We show that (in, out) is a zone of $d_1 + (d_2 + d_3)$, that is $(in, out) \in Z_1 + (Z_2 + Z_3)$. By Lemma 11, Part 1, we deduce that

- $(in \cap L_1, out \cap L_1) \in Z_1$,
- $(in \cap L_2, out \cap L_2) \in Z_2$, and
- $(in \cap L_3, out \cap L_3) \in Z_3$.

Define
$$in_z = (in \cap L_2) \cup (in \cap L_3) = in \cap (L_2 \cup L_3)$$
and
$$out_z = (out \cap L_2) \cup (out \cap L_3) = out \cap (L_2 \cup L_3).$$
Then
$$\begin{aligned} in_z \cup out_z &= (in \cap (L_2 \cup L_3)) \cup (out \cap (L_2 \cup L_3)) \\ &= (in \cup out) \cap (L_2 \cup L_3) \\ &= L_2 \cup L_3 \qquad \text{since } L_2 \cup L_3 \subseteq in \cup out. \end{aligned}$$

Therefore, (in_z, out_z) is a zone where:

(a) $in_z \cup out_z = L_2 \cup L_3$,
(b) $(in_z \cap L_2, out_z \cap L_2) \in Z_2$ and
(c) $(in_z \cap L_3, out_z \cap L_3) \in Z_3$.

By Lemma 11, Part 2, we deduce that $(in_z, out_z) \in Z_2 + Z_3$. Now we have (in, out) is a zone where:

(a) $in \cup out = L_1 \cup (L_2 \cup L_3)$,
(b) $(in \cap L_1, out \cap L_1) \in Z_1$ and
(c) $(in_z, out_z) = (in \cap (L_2 \cup L_3), out \cap (L_2 \cup L_3)) \in Z_2 + Z_3$.

By Lemma 11, Part 2, we deduce that $(in, out) \in Z_1 + (Z_2 + Z_3)$. Therefore $(Z_1 + Z_2) + Z_3 \subseteq Z_1 + (Z_2 + Z_3)$. The proof that $Z_1 + (Z_2 + Z_3) \subseteq (Z_1 + Z_2) + Z_3$ is similar. Therefore, $(d_1 + d_2) + d_3 = d_1 + (d_2 + d_3)$. Hence + is associative, as required. □

Lemma 10 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\mathcal{S}$ and $\{d_{\mathcal{S}}\}$ are semantically equivalent.*

Proof By Lemma 4, we know that for each s in $\mathcal{S}$, s semantically entails d, where $s \rightsquigarrow d$. Given such as s and d, we now show that d semantically entails s. Let $\mathcal{I} = (\triangle, \Psi)$ be a model for d. There are two cases.

- Case 1: $s \equiv s_1 \subseteq s_2$. In this case, we know that

$$MZ(d) = \mathcal{T}(s_1, L(d)) \backslash \mathcal{T}(s_2, L(d)).$$

Since $\mathcal{I}$ is a model for d, it is the case that

$$\Psi(\mathcal{T}(s_1, L(d)) \backslash \mathcal{T}(s_2, L(d))) = \emptyset.$$

This implies

$$\Psi(\mathcal{T}(s_1, L(d))) \backslash \Psi(\mathcal{T}(s_2, L(d))) = \emptyset$$

from which it follows that

$$\Psi(\mathcal{T}(s_1, L(d))) \subseteq \Psi(\mathcal{T}(s_2, L(d))).$$

By Lemma 3, $\Psi(s_1) = \Psi(\mathcal{T}(s_1, L(d)))$ and $\Psi(s_2) = \Psi(\mathcal{T}(s_2, L(d)))$, so $\Psi(s_1) \subseteq \Psi(s_2)$. Therefore $\mathcal{I}$ models $s \equiv s_1 \subseteq s_2$.
- Case 2: $s \equiv s_1 = s_2$. Noting that $s_1 = s_2$ is equivalent to $s_1 \subseteq s_2$ and $s_2 \subseteq s_1$, this case is similar to Case 1.

In both cases, we have shown that $\mathcal{I}$ models s, so d semantically entails s. Therefore s and d are semantically equivalent. Thus, given $\mathcal{S} = \{s_1, ..., s_n\}$ and the set of diagrams, $\mathcal{D} = \{d_1, ..., d_n\}$ observable from the set-theoretic sentences in $\mathcal{S}$, we know that $\mathcal{D}$ is semantically equivalent to $\mathcal{S}$. By Lemma 8, $\{d_1 + ... + d_n\} = \{d_\mathcal{S}\}$ is semantically equivalent to $\mathcal{D}$. Hence $\mathcal{S}$ is semantically equivalent to $\{d_\mathcal{S}\}$. □

B Proofs of the Theorems

Theorem 1 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\mathcal{S}$ is observationally devoid with respect to $\mathcal{S}_\vDash$.*

Proof Let s be in $\mathcal{S}_\vDash$. Then s is properly semantically entailed by $\mathcal{S}$. This means that s is not in $\mathcal{S}$. The only statements observable from $\mathcal{S}$ are those in $\mathcal{S}$. Therefore s is not in $\mathcal{O}(\mathcal{S})$. Hence $\mathcal{S}$ is observationally devoid with respect to $\mathcal{S}_\vDash$. That is, $\mathcal{S}_\vDash \cap \mathcal{O}(\mathcal{S}) = \emptyset$ as required by Definition 3. □

Theorem 2 *Let $\mathcal{S}$ be a finite set of set-theoretic sentences. Then $\{d_\mathcal{S}\}$ is observationally complete with respect to $\mathcal{S}_\vDash^L$ where $L = \bigcup_{s \in \mathcal{S}} L(s)$.*

Proof We start by showing that for each set-expression, s, where $L(s) \subseteq L = L(d_\mathcal{S})$, there is a region in $d_\mathcal{S}$ that translates to s. This part of the proof is by induction on the depth of s in the inductive construction of set-expressions. There are three base cases:

- Case 1: $s \equiv \emptyset$. Set $r = \emptyset$ and we are done.

- Case 2: $s \equiv U$. Here, take $r = Z(d_{\mathcal{S}})$. In this case,

$$r = \{(in, out) \in Z(d_{\mathcal{S}}) : \emptyset \subseteq in \wedge \emptyset \subseteq out\}.$$

  Therefore, r is the zonal region $\langle \emptyset, \emptyset \rangle$. Therefore r has description $\langle \emptyset, \emptyset \rangle$ and, so, translates to U as required.
- Case 3: $s \equiv l$ for some $l \in \mathcal{L}$. Take r to be the largest set of zones in $Z(d_{\mathcal{S}})$ such that, for each (in, out) in r, $l \in in$. That is

$$\begin{aligned} r &= \{(in, out) \in Z(d_{\mathcal{S}}) : l \in in\} \\ &= \{(in, out) \in Z(d_{\mathcal{S}}) : \{l\} \subseteq in \wedge \emptyset \subseteq out\}. \end{aligned}$$

  Then, since $l \in L(d_{\mathcal{S}})$, r has description $\langle \{l\}, \emptyset \rangle$ and, so, translates to $s \equiv l$.

Assume for all set-expressions, s, at depth n in the inductive construction, where $L(s) \subseteq L(d_{\mathcal{S}})$ that such an r exists. Let s' be a set-expression at depth $n+1$ where $L(s') \subseteq L(d_{\mathcal{S}})$. Then either $s' \equiv (s_1 \star s_2)$, where $\star \in \{\cap, \cup, \setminus\}$, or $s' \equiv \overline{s_1}$, for some set-expressions s_1 and s_2 at depth n. By assumption, there are regions r_1 and r_2 in d that translate to s_1 and s_2 respectively. We have the following cases:

- Case i: $s \equiv (s_1 \star s_2)$. By the inductive assumption, for the set-expressions s_1 there exists a region, r_1 in $d_{\mathcal{S}}$ that translates to s_1. Similarly, such an r_2 exists for s_2. Further, we know that r_1 and r_2 translate to s_1 and s_2 via some descriptions, say R_1 and R_2 of r_1 and r_2 respectively. Since the set of regions in any diagram is closed under union and intersection, the region $r = r_1 \star r_2$ is in $d_{\mathcal{S}}$. By Definition 29, the region r has description $R_1 \star R_2$ and, using $R_1 \star R_2$ and our inductive assumption, we see that r translates to s via $R_1 \star R_2$ as required.
- Case ii: $s \equiv \overline{s_1}$. Noting that set of regions in any diagram is closed under complement, the argument is similar to Case i.

Therefore, every set-expression, s, where $L(s) \subseteq L(d_{\mathcal{S}})$ has a region, r, in $d_{\mathcal{S}}$ translates to s.

For the next part of the proof, let s be a set-theoretic sentence that is in $\mathcal{S}^{L}_{\models}$. We must show that s is observable from $d_{\mathcal{S}}$. By the construction of $d_{\mathcal{S}}$, we first note that $L = L(d_{\mathcal{S}})$. Let $\mathcal{I}$ be a model for $d_{\mathcal{S}}$. By Lemma 10, $d_{\mathcal{S}}$ semantically entails s so $\mathcal{I}$ also models s. Now, either $s \equiv s_1 \subseteq s_2$ or $s \equiv s_1 = s_2$. We consider these two cases.

- Case A: $s \equiv s_1 \subseteq s_2$. We have just seen that there are regions, r_1 and r_2 in $d_{\mathcal{S}}$ that translate to s_1 and s_2 respectively. We show that $r_1 \subseteq r_2$. Since $\mathcal{I}$ models $s_1 \subseteq s_2$, we know that $\Psi(s_1) \subseteq \Psi(s_2)$. By Lemma 6, this implies that $\Psi(r_1) \subseteq \Psi(r_2)$. If there were a zone, z, in r_1 that is not in r_2 then there would be a model for $d_{\mathcal{S}}$ in which $\Psi(r_1) \not\subseteq \Psi(r_2)$. But such a model for $d_{\mathcal{S}}$ would not satisfy $s_1 \subseteq s_2$. Therefore no such zone can exist. Thus, $r_1 \subseteq r_2$. We have:
  1. $r_1 \subseteq r_2$,
  2. s_1 is a translation of r_1, and

3. s_2 is a translation of r_2.
Therefore, $s_1 \subseteq r_2$ is observable from $d_{\mathcal{S}}$.

– Case B: $s \equiv s_1 = s_2$. Noting that $s_1 = s_2$ is semantically equivalent to $s_1 \subseteq s_2$ and $s_2 \subseteq s_1$, the proof is similar to Case A.

Hence, in either case, s is observable from $d_{\mathcal{S}}$. Therefore, every set-theoretic sentence in $\mathcal{S}^L_{\models}$ is observable from $d_{\mathcal{S}}$. Hence, by Definition 2, $d_{\mathcal{S}}$ is observationally complete with respect to $\mathcal{S}^L_{\models}$. □

Theorem 3 *Let* $\mathcal{S} = \{s_1, ..., s_n\}$ *be a finite set of set-theoretic sentences. Then* $\mathcal{D} = \{d_{\mathcal{S}}\}$ *has maximal observational advantage over* $\mathcal{S}$ *given* $\mathcal{S}^L_{\models}$ *where* $L = \bigcup_{s \in \mathcal{S}} L(s)$.

Proof By Lemma 10, $\mathcal{S}$ and $\{d_{\mathcal{S}}\}$ are semantically equivalent. By Theorem 1, $\mathcal{S}$ is observationally devoid with respect to $\mathcal{S}_{\models}$. Since $\mathcal{S}^L_{\models} \subseteq \mathcal{S}_{\models}$, $\mathcal{S}$ is also observationally devoid with respect to $\mathcal{S}^L_{\models}$. By Theorem 2, $\{d_{\mathcal{S}}\}$ is observationally complete with respect to $\mathcal{S}^L_{\models}$. Therefore, $\{d_{\mathcal{S}}\}$ has maximal observational advantage over $\mathcal{S}$ given $\mathcal{S}^L_{\models}$. □