Classifying links under fused isotopy

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ABSTRACT

All knots have been shown to be isotopic to the unknot using a process known as virtualization. We extend and adapt this process to show that, up to fused isotopy, classical links are classified by their linking numbers. We provide an algebraic proof, utilising Alexander's Theorem and some simple results about the pure braid group.

Keywords: Virtual and fused braids, virtual and fused links, linking number, fused isotopy

Mathematics Subject Classification 2010: 57M25

1. Introduction

Classical braids and links have been generalized to the virtual category [10] – adding virtual crossings and extending isotopy to allow the virtual analogues of the classical Reidemeister moves. The forbidden moves F_o and F_u, shown in Fig. 1, are not allowable under virtual isotopy. Extending virtual isotopy in the virtual braid group VB_n to allow the F_o move gives rise to the welded braid group WB_n, which has been shown to be isomorphic to PC_n, the group of automorphisms of the free group on n elements of permutation-conjugacy type [4]. Allowing both of the forbidden moves F_o and F_u gives rise to fused isotopy [10]. That is, two virtual links L_1 and L_2 are called *fused isotopic* if L_2 can be obtained from L_1 by a finite sequence of Reidemeister moves, virtual moves and F_o, F_u moves.

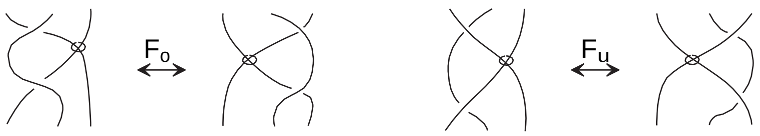

Fig. 1. The forbidden moves

Let K be the space of classical links embedded in S^3 and let VK be the space of virtual links. Kauffman [10], and independently Goussarov-Polyak-Viro [7], have shown that K embeds into VK. Let f denote the natural inclusion of VK into the space of fused links

FK. Then we have $K \overset{\mathbf{i}}{\hookrightarrow} VK \overset{\mathbf{f}}{\to} FK$, and when we refer to a classical link (under fused isotopy) we mean $\mathbf{f} \circ \mathbf{i}(L)$, the image of a link $L \in K$ in the space FK.

In [9], Kanenobu showed that all knots are fused isotopic to the unknot. He showed that all of the classical crossings of a virtual knot can be *virtualized*; that is every classical crossing can be changed into a virtual crossing by applying a sequence of fused isotopy moves. However, crossings between different components of a link cannot be virtualized using the same methods. The following theorem from [9] provides us with allowable moves under fused isotopy which were used in the virtualization procedure.

Theorem 1. The moves M_1, M_2 and M_3, shown in Fig. 2, can be realised by fused isotopy.

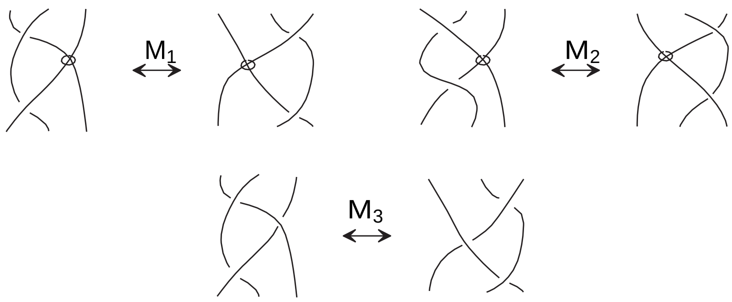

Fig. 2. Allowable moves in fused isotopy

In [5] the authors showed that the Jones polynomial for welded and fused links is well-defined in a quotient of $Z[A, A^{-1}]$ and observed that this polynomial depends only upon the linking number for links with two components. Inspired by this, we show that classical links, under fused isotopy, can be determined by the linking number of their components.

Theorem 2. A classical link L with n–components is completely determined by the linking numbers of each pair of components under fused isotopy.

The strategy that we use to prove Theorem 2 is to write L as the closure of a braid α on m strands (where $m \geq n$) and then to transform α into a pure braid β on n strands whose closure is also L. We show that β depends only on the linking numbers of the components of L. This means that any classical link with the same linking numbers as L can be obtained as the closure of β. We need some preliminaries before we proceed with the proof.

2. Preliminaries

Recall that an element of the pure braid group P_n is an n–strand braid where the permutation induced by the strings is the identity. The pure braid group P_n has a presentation with generators $A_{\mathbf{i},\mathbf{j}}$ with $1 \leq \mathbf{i} < \mathbf{j} \leq n$ where

$$A_{\mathbf{i},\mathbf{j}} = \sigma_{\mathbf{j}-1}\sigma_{\mathbf{j}-2}\dots\sigma_{\mathbf{i}}^2\dots\sigma_{\mathbf{j}-2}^{-1}\sigma_{\mathbf{j}-1}^{-1} = \sigma_{\mathbf{i}}^{-1}\sigma_{\mathbf{i}+1}^{-1}\dots\sigma_{\mathbf{j}-1}^2\dots\sigma_{\mathbf{i}+1}\sigma_{\mathbf{i}}.$$

In particular, we have $A_{j,j+1} = \sigma_j^2$. Let U_k be the subgroup of P_n generated by $\{A_{i,k} : 1 \leq i < k\}$. Then every element of P_n can be written in the unique normal form $x_2 x_3 \dots x_n$, where $x_k \in U_k$ (see [2] for details). Define $B_{i,j} := \sigma_{j-1} \dots \sigma_{i+1}\sigma_i$ for $i < j$ and $B_{i,i} := 1$. Then by definition $A_{i,j+1} = B_{i,j}^{-1} A_{j,j+1} B_{i,j}$, and we can see from Fig. 3 that for $k < i < j$, $B_{i,j}$ commutes with $A_{k,j+1}$ in B_n.

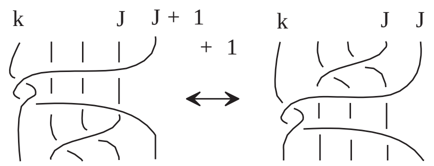

Fig. 3. $A_{k,j+1} B_{i,j} = B_{i,j} A_{k,j+1}$

The virtual braid group on n–strands, VB_n, can be defined by adding extra generators τ_i, for $1 \leq i \leq n$, corresponding to the virtual crossings, and relations corresponding to the virtual isotopy moves [8,10].

We define the *fused braid group*, FB_n by adding the relations $\sigma_i^{-1}\tau_j\sigma_i = \sigma_j\tau_i\sigma_j^{-1}$ with $|i - j| = 1$, to the virtual braid group VB_n. If $j = i + 1$ the relation corresponds to the F_u move, and if $i = j + 1$ it corresponds to the F_o move. The *fused pure braid group*, FP_n, is the group of fused braids for which the induced permutation is the identity.

The explicit realization of the moves M_1, M_2 and M_3 using F_o and F_u moves is shown in [9], and this gives rise to the following consequences in FB_n:

$$\begin{aligned} M_1 &: \quad \sigma_i\tau_j\sigma_i = \sigma_j\tau_i\sigma_j \\ M_2 &: \sigma_i^{-1}\tau_j\sigma_i^{-1} = \sigma_j^{-1}\tau_i\sigma_j^{-1} \\ M_3 &: \quad \sigma_i\sigma_j^{-1}\sigma_i = \sigma_j\sigma_i^{-1}\sigma_j, \end{aligned} \tag{2.1}$$

where $|i - j| = 1$.

The following lemmas are used in the proof of Theorem 2, and the indices have been chosen to match the usage in the proof. Let $\sim$ denote the equivalence class generated by fused isotopy and let $\mho_k = U_k / \sim$. In the following, we refer to application of the braid relations $\sigma_i^{\varrho}\sigma_j^{\delta} = \sigma_j^{\delta}\sigma_i^{\varrho}$ for $|i - j| > 1$, and $\varrho, \delta \in \{1, -1\}$, as *commutations in* B_n, and the use of the virtual braid relations $\sigma_i^{\varrho}\tau_j^{\delta} = \tau_j^{\delta}\sigma_i^{\varrho}$, for $|i - j| > 1$, and $\varrho, \delta \in \{1, -1\}$, as *commutations in* VB_n.

Lemma 0.1. In the fused pure braid group FP_n we have:

$$A_{j,j+1} A_{i,j+1} A_{j,j+1}^{-1} = A_{i,j+1} \text{ where } 1 \leq i < j + 1 \leq n. \tag{2.2}$$

In other words, $A_{j,j+1} = \sigma_j^2$ is in the centre of $\mho_{j+1}$.

Proof. Using the relations $\sigma_j\sigma_{j-1}^{-1}\sigma_j = \sigma_{j-1}\sigma_j^{-1}\sigma_{j-1}$ corresponding to an M_3 move, and $\sigma_j\sigma_{j-1}\sigma_j^{-1} = \sigma_{j-1}^{-1}\sigma_j\sigma_{j-1}$ corresponding to an R_3 move, we obtain

$$\begin{aligned}\sigma_j^2\sigma_{j-1}^{-1}\sigma_j^2\sigma_{j-1}\sigma_j^{-2} &= \sigma_j\ \ \sigma_j\sigma_{j-1}^{-1}\sigma_j\ \ \ \sigma_j\sigma_{j-1}\sigma_j^{-1}\ \ \sigma_j^{-1}\\ &= \sigma_j\ \ \sigma_{j-1}\sigma_j^{-1}\sigma_{j-1}\ \ \ \sigma_{j-1}^{-1}\sigma_j\sigma_{j-1}\ \ \sigma_j^{-1}\\ &= \sigma_j\sigma_{j-1}^2\sigma_j^{-1}.\end{aligned} \tag{2.3}$$

Using Eq. (2.3), we can compute

$$\begin{aligned}&A_{j,j+1}A_{i,j+1}A_{j,j+1}^{-1}\\ &= \sigma_j^2B_{i,j}^{-1}A_{j,j+1}B_{i,j}\sigma_j^{-2}\\ &= \sigma_j^2(\sigma_i^{-1}\dots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1})\sigma_j^2(\sigma_{j-1}\sigma_{j-2}\dots\sigma_i)\sigma_j^{-2} && \text{by definition}\\ &= (\sigma_i^{-1}\dots\sigma_{j-2}^{-1})\sigma_j^2\sigma_{j-1}^{-1}\sigma_j^2\sigma_{j-1}\sigma_j^{-2}(\sigma_{j-2}\dots\sigma_i) && \text{by commutation in } B_n\\ &= B_{i,j-1}^{-1}\sigma_j^2\sigma_{j-1}^{-1}\sigma_j^2\sigma_{j-1}\sigma_j^{-2}B_{i,j-1} && \text{by definition}\\ &= B_{i,j-1}^{-1}\sigma_j\sigma_{j-1}^2\sigma_j^{-1}B_{i,j-1} && \text{by Eq. (2.3)}\\ &= (\sigma_i^{-1}\dots\sigma_{j-2}^{-1})\sigma_j\sigma_{j-1}^2\sigma_j^{-1}(\sigma_{j-2}\dots\sigma_i) && \text{by definition}\\ &= \sigma_j(\sigma_i^{-1}\dots\sigma_{j-2}^{-1})\sigma_{j-1}^2(\sigma_{j-2}\dots\sigma_i)\sigma_j^{-1} && \text{by commutation in } B_n\\ &= \sigma_jB_{i,j-1}^{-1}\sigma_{j-1}^2B_{i,j-1}\sigma_j^{-1} && \text{by definition}\\ &= \sigma_jA_{i,j}\sigma_j^{-1}\\ &= A_{i,j+1}.\end{aligned}$$

□

Lemma 0.2. For every $1 \le j+1 < n$, the subgroup U_{j+1} of FP_n is commutative.

Proof. Assume without loss of generality that $k < i$. Then

$$\begin{aligned}A_{k,j+1}A_{i,j+1} &= A_{k,j+1}B_{i,j}^{-1}A_{j,j+1}B_{i,j}\\ &= B_{i,j}^{-1}A_{k,j+1}A_{j,j+1}B_{i,j} && \text{by commutation in } B_n\\ &= B_{i,j}^{-1}A_{j,j+1}A_{k,j+1}B_{i,j} && \text{by Lemma 0.1}\\ &= B_{i,j}^{-1}A_{j,j+1}B_{i,j}A_{k,j+1} && \text{by commutation in } B_n\\ &= A_{i,j+1}A_{k,j+1}.\end{aligned}$$

□

Lemma 0.3. In FB_n we have:

$$A_{i,j+1}\tau_j = \tau_jA_{i,j} \qquad \text{for } 1 \le i \le j-1. \tag{2.4}$$

Proof. Using the relations $\sigma_j\sigma_{j-1}\tau_j = \tau_{j-1}\sigma_j\sigma_{j-1}$ corresponding to an F_o move, and $\sigma_{j-1}^{-1}\sigma_j\tau_{j-1} = \tau_j\sigma_{j-1}\sigma_j^{-1}$ corresponding to an M_1 move, we obtain

$$\begin{aligned}A_{i,j+1}\tau_j &= B_{i,j}^{-1}\sigma_j^2B_{i,j}\tau_j\\ &= B_{i,j-1}^{-1}\sigma_{j-1}^{-1}\sigma_j\sigma_j\sigma_{j-1}B_{i,j-1}\tau_j\\ &= B_{i,j-1}^{-1}\sigma_{j-1}^{-1}\sigma_j\sigma_j\sigma_{j-1}\tau_jB_{i,j-1} && \text{by commutation in } VB_n\\ &= B_{i,j-1}^{-1}\sigma_{j-1}^{-1}\sigma_j\tau_{j-1}\sigma_j\sigma_{j-1}B_{i,j-1} && \text{by an } F_o \text{ move}\\ &= B_{i,j-1}^{-1}\tau_j\sigma_{j-1}\sigma_j^{-1}\sigma_j\sigma_{j-1}B_{i,j-1} && \text{by an } M_1 \text{ move}\\ &= \tau_jB_{i,j-1}^{-1}\sigma_{j-1}^2B_{i,j-1} && \text{by commutation in } VB_n\\ &= \tau_jA_{i,j}.\end{aligned}$$

□

3. Proof of Theorem 2

Let L be a classical link with n–components. By Alexander's Theorem, there exists $\alpha \in B_m$ with $m \geq n$, such that the closure of α is L. Chow [3] (or see page 22 of [2]) shows that every α can be written in the form $\alpha = x_2 B_{k_2,2} \ldots x_m B_{k_m,m}$ where $x_i \in U_i \leq P_m$ and $1 \leq k_i \leq i$. Let $\hat{\alpha}$ denote the closure of α. If $m > n$ then we will construct $\beta \in B_n$ such that $\hat{\beta}$ is fused isotopic to $\hat{\alpha}$.

If $B_{k_i,i} = 1$ for all $i = 2, \ldots, m$ then α is a pure braid, which means that m must be equal to n. So let us assume that $B_{k_s,s} \neq 1$, for some s, and that if $i > s$ then $B_{k_i,i} = 1$. This means that the permutation induced by α is the identity on the strands $s + 1, \ldots, m$. Therefore, each of these strands forms a separate component of the link $\hat{\alpha}$. Now, conjugating α with $B_{1,m}$ gives $B_{1,m}^{-1} \alpha B_{1,m}$, and as shown in Fig. 4, the $(m-1)$-st strand of the original braid α becomes the m-th strand of the new braid.

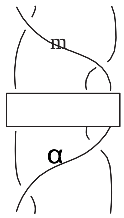

Fig. 4. $B_{1,m}^{-1} \alpha B_{1,m}$

Thus if we conjugate α with $B_{1,m}$ $(m - s)$ times, we get $\alpha' = B_{1,m}^{s-m} \alpha B_{1,m}^{m-s}$ and the s-th strand of α becomes the m-th strand of α'. Since α' is just a conjugate of α their closures are isotopic. Now write $\alpha' = y_2 B_{t_2,2} \ldots y_m B_{t_m,m}$ with $y_i \in U_i$. Then $B_{t_m,m} \neq 1$ and so by definition, $B_{t_m,m} = \sigma_{m-1} B_{t_m,m-1}$. A picture of α' is shown in Fig. 5, where $W = y_2 B_{t_2,2} \ldots y_{m-1} B_{t_{m-1},m-1}$.

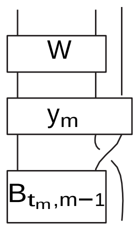

Fig. 5. The braid α'

Since $\mho_m$ is commutative (by Lemma 0.2), we can write

$$y_m = A^{r_1}_{1,m} \dots A^{r_{m-2}}_{m-2,m} \underbrace{A^{r_{m-1}}_{m-1,m}}_{\sigma^{2r_{m-1}}_{m-1}} \quad \text{for some } r_1, \dots, r_{m-1}.$$

By definition, $A^{r_{m-1}}_{m-1,m} = \sigma^{2r_{m-1}}_{m-1}$, and since $B_{t_m,m} = \sigma_{m-1} B_{t_m,m-1}$, we obtain

$$y_m B_{t_m,m} = A^{r_1}_{1,m} \dots A^{r_{m-2}}_{m-2,m} \sigma^{2r_{m-1}+1}_{m-1} B_{t_m,m-1}.$$

Since W does not involve the m-th strand and y_m is a pure braid, Fig. 5 shows that the m-th strand and the other strand that is involved in the last occurrence (and hence in all of the previous occurrences) of σ_{m-1} in α' belong to the same component of $L = \hat{\alpha}'$. Therefore, following the strategy in [9], we can virtualize all of the $2r_{m-1} + 1$ crossings in $\hat{\alpha}'$ which correspond to $\sigma^{2r_{m-1}+1}_{m-1}$ in α'. In doing so we have not changed the fused isotopy class of L but we have obtained L as the closure of

$$\begin{aligned}\alpha_1 &= W A^{r_1}_{1,m} \dots A^{r_{m-2}}_{m-2,m} \tau^{2r_{m-1}+1}_{m-1} B_{t_m,m-1} \\ &= W A^{r_1}_{1,m} \dots A^{r_{m-2}}_{m-2,m} \tau_{m-1} B_{t_m,m-1}.\end{aligned}$$

By Lemma 0.3, we obtain

$$\alpha_1 = W \tau_{m-1} v_{m-1} B_{t_m,m-1}$$

where $v_{m-1} = A^{r_1}_{1,m-1} \dots A^{r_{m-2}}_{m-2,m-1}$ which is an element of $\mho_{m-1}$.

Figure 6 shows that there is only one crossing involving the m-th strand in the braid α_1. This is the occurrence of τ_{m-1}. In $\hat{\alpha}_1$, we can get rid of the virtual crossing corresponding to τ_{m-1} with a virtual move (of type I). We have obtained a new link diagram $\hat{\alpha}_2$ where

$$\alpha_2 = W v_{m-1} B_{t_m,m-1}$$

has $m - 1$ strands and $\hat{\alpha}_2$ is fused isotopic to L.

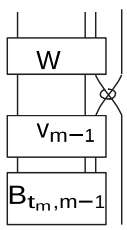

Fig. 6. The braid α_1

If we continue with this process, eventually we will get a braid β in B_n whose closure is fused isotopic to L. Note that since β has n strands and L has n components, each strand of β corresponds to a different component of L and therefore β must be a pure braid. For

$i < j$, define the group homomorphism $\delta_{i,j} : PB_n \to \mathbb{Z}$ by

$$\delta_{i,j}(A_{s,t}) = \begin{cases} 1 & \text{if } s = i \text{ and } t = j \\ 0 & \text{otherwise.} \end{cases}$$

Since β is a pure braid it is easy to see that $\delta_{i,j}(\beta) = lk(\ell_i, \ell_j)$ where ℓ_i and ℓ_j are the corresponding components of $\hat{\beta}$.

This proves that any classical link L with n–components can be obtained as the closure of a pure braid $\beta = x_2 \dots x_n$ and since U_k is commutative for every k, each x_k can be written in the form $x_k = A_{1,k}^{\delta_{1,k}} \dots A_{k-1,k}^{\delta_{k-1,k}}$ where $\delta_{i,k}$ denotes $\delta_{i,k}(\beta)$. This shows that β depends only on the linking number of the components. □

4. Discussion

Theorem 2 does not immediately generalize to non-classical links where there are virtual crossings between different components. For example, let U_2 be the trivial link with two components and let $L = \hat{\alpha}$ where $\alpha = \sigma_1\tau_1\sigma_1^{-1}\tau_1 \in FB_2$. Both of these links have (classical) linking number zero, but they are not fused isotopic. In [1] they point out that they can be distinguished by the refinement of the classical linking number via the pair of (virtual) linking numbers of [7]. This is given by ordering the two components and then: (i) counting the sum of the signs of the classical crossing for which component 1 goes over component 2, (ii) counting the sum of the signs of the classical crossing for which component 2 goes over component 1; the classical linking number is then recoverable from this pair.

The work in this paper was originally archived in [6], and was produced independently of the work in [11], which the authors have only recently managed to obtain a copy. In [11], a geometric argument is provided for the classification of (ordered) oriented fused links via their linking numbers and their virtual linking numbers as defined in [11]; these are different to the virtual linking numbers considered in [7], instead taking a sum of signs over the virtual crossings, with sign of virtual crossings being determined by their order of appearance, and is dependent upon the choice of ordering of the components. The geometric proof in [11] makes use of specific constructions, such as the use of "fusions of virtual Hopf links". In contrast, in this paper, we provide an algebraic proof of the classification of (unordered) oriented classical links under fused isotopy, with recourse to standard machinery of Alexander's Theorem, and the theory of braids. Along the way we provide some additional simple results about braids required for the main construction. In the case of classical links, the virtual linking numbers, in the sense of [11], would be zero and the results obtained about the classification are consistent.

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